Dade's conjecture for the McLaughlin Simple Group

John C. Murray

Abstract

In [**Da92**] E.C. Dade announced the first of what was to be a series of conjectures concerned with counting characters in the blocks of finite groups. Specifically, Dade's so-called Ordinary Conjecture asserts that if a finite group G has trivial p-core, and B is a p-block of G of positive defect, then the number $k(B, d)$ of complex irreducible characters with fixed defect d belonging to B can be expressed as an alternating sum over the corresponding numbers for the normalizers of the non-trivial p-chains of G. Refined versions of this conjecture have been given in subsequent articles. See [**Da94**] and [**DaPr**] for further details. Dade claims that the strongest form of his conjectures is true for all finite groups if it is true for all covering groups of finite simple groups. Thus, in order to prove his conjectures, one merely has to go through the complete list of finite simple groups, carefully verifying that the strongest form holds for each one. In this Thesis we prove the most general version of Dade's Conjectures for all covering groups of the McLaughlin group $\mathbf{M_c}$. This is one of the 26 sporadic simple groups. So our Thesis forms a portion of the as yet incomplete proof of Dade's Conjectures for all finite groups.

Dedication

To my wife Deirdre and our children Clara and Seóirse.

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Chapter 1

Introduction

In [**Da92**] E.C. Dade announced his "Ordinary Conjecture" which asserts that if a finite group G has trivial p-core, and B is a p-block of G of positive defect, then the number $k(B, d)$ of complex irreducible characters with fixed defect d belonging to B can be expressed as an alternating sum, as in 1.4.2, over the corresponding numbers for the normalizers of the non-trivial p-chains of G. In subsequent articles [**Da94**] and [**DaPr**] he gave stronger versions of this conjecture. He has announced that the strongest form of his conjectures, the so-called "Inductive Conjecture", will have the property that it is true for all finite groups if it is true for all covering groups of finite simple groups.

The aim of this thesis is to verify that the strongest form of Dade's conjectures holds for all covering groups of the McLaughlin Simple Group $\mathbf{M_c}$, for all primes p. We use some basic facts about $\mathbf{M_c}$ obtained from the Atlas [**Con85**] to simplify our task. The group $\mathbf{M_c}$ has exactly two covering groups: $\mathbf{M_c}$ and $3.\mathbf{M_c}$. The outer automorphism group $\mathrm{Out}(\mathbf{M_c})$ of $\mathbf{M_c}$ is cyclic of order 2. In this situation Dade asserts in [**DaPr**] that the Inductive Conjecture for $\mathbf{M_c}$ is equivalent to the weaker "Invariant Conjecture" as outlined in 1.4.4. Furthermore, Dade has proved in [**Da96**] that this Invariant Conjecture is true for all blocks with cyclic defect groups. The group $\mathrm{Out}(3.\mathbf{M_c} \mid \mathbf{A})$ of outer automorphisms of $3.\mathbf{M_c}$ centralizing $\mathbf{A} = \mathrm{Z}(3.\mathbf{M_c})$ is trivial.

In this situation Dade asserts in [**DaPr**] that the Inductive Conjecture is equivalent to the "Projective Conjecture", as outlined in 1.4.6. Moreover, this weak form of the Inductive Conjecture holds for blocks with cyclic defect groups, as shown in [**Da96**]. From the Atlas $|\mathbf{M_c}| = 2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$. For both covering groups we need therefore only worry about the primes $p = 5, 3, 2$.

In the Introduction we outline the theory needed to understand the conjectures and the methods we will use. The rest of the Thesis consists of three chapters; one for each of the primes $p = 5, 3, 2$. We deal with $p = 5$ first, as it is the easiest case. It turns out that the 3-local structure is useful for determining the 2-local structure, so we deal with $p = 3$ before $p = 2$. In all cases our first task is to obtain enough information on the p-local structure to be able to determine the conjugacy classes of radical p subgroups of $\mathbf{M_c}$ and their normalizers in $\mathbf{M_c}$ and $\mathbf{M_c}.2$. This in turn enables us to determine the conjugacy classes of radical p-chains and to describe their normalizers. Finally we use Clifford Theory to obtain the character degrees of the the chain normalizers. The Invariant Conjecture for $\mathbf{M_c}$ and $p = 5$ follows from Theorems 2.4.2 and 2.6.2, while the Projective Conjecture for $3.\mathbf{M_c}$ and $p = 5$ follows from Theorem 2.7.1; for $p = 3$ the corresponding Theorems are 3.6.2, 3.11.2 and 3.16.2; while for $p = 2$ the relevant Theorems are 4.8.3, 4.9.19 and 4.10.12. These results are enough to prove that the Inductive Conjecture holds for all covering groups of $\mathbf{M_c}$.

A final word. The philosophy behind Dade's conjectures is that one can obtain information on the global block structure of a finite group G from knowledge of the local block structure. The group $3.\mathbf{M_c}$ is best viewed as a certain maximal subgroup of Conway's Group **Co**. However, it turned out that in order to obtain the information needed to prove the results in our thesis, it was enough to consult the Atlas as well as obtain information on the local structure of $\mathbf{M_c}$ from [**JW72**] and [**Fk73**]. Thus the methods of proof found here are in accord with the local-to-global approach promoted by these conjectures.

1.1. Notation

Due to the limited number of symbols available, our notation may vary between chapters. We will try to warn the reader whenever this occurs.

Some of our notation is nonstandard. In particular if G is a finite group and X is a G-set, then $\mathrm{Orb}(G, X)$ will denote the multiset of orbit lengths of G on X, rather than the set of orbits of G on X. We will often deal with the situation where G has normal subgroups X and Y, with $Y \leq X$. Then $\mathrm{Irr}(G)$ will denote the set of ordinary irreducible characters of G, while $\mathrm{Irr}(G \mid X)$ will denote those elements of $\mathrm{Irr}(G)$ which have non-trivial restriction to X. The set $\mathrm{Irr}(G \mid G)$ will be denoted by the more usual $\mathrm{Irr}(G)^{\times}$. When we wish to refer to the multiset of the degrees of those irreducible characters of G which are nontrivial on X but trivial on Y, we use $\mathrm{Deg}(G \mid X/Y)$. If now G is embedded as a normal subgroup of some finite group E, while X and Y are normal subgroups of E, and not just G, then $\mathrm{Inv}(G \mid X/Y)$ will stand for the multiset of degrees of those irreducible characters of G which are invariant in E, and non-trivial on X but trivial on Y. The group Y is omitted if it is $\{1\}$. As would be expected, $\mathrm{Deg}(G/X)$ or $\mathrm{Inv}(G/X)$ refer to the characters of the factor group G/X. We use $\mathrm{Deg}(G \bmod X)$ and $\mathrm{Inv}(G \bmod X)$ to refer to the characters of G inflated from G/X.

The p-defect of an irreducible character $\chi \in \mathrm{Irr}(G)$ is the largest power of p which divides $|G|/\chi(1)$. We use $\mathrm{Def}_p(G)$ to refer to the multiset of p-defects of the irreducible characters of G. If the p is omitted, the context should make clear what prime we are using. We use $\mathrm{InvDef}_p(G)$ to refer to the multiset of p-defects of the irreducible characters of G which are invariant in the extension E of G. The group E will be clear from the context. We can extend these notations by using $\mathrm{Def}_p(G \mid X/Y)$ and $\mathrm{InvDef}_p(G \mid X/Y)$, when referring to characters of G nontrivial on X but trivial on Y. We use $\mathrm{Def}_p(G \bmod X)$ and $\mathrm{InvDef}_p(G \bmod X)$ to refer to all the characters of G which are inflated from G/X. In particular, the defects are measured using G rather than G/X.

In all cases, if we replace the normal subgroup X of G by an irreducible character ξ of X, then we will be referring to characters of G lying over ξ. For instance $\mathrm{Deg}(G \mid \xi)$ is the multiset of the degrees of those irreducible characters of G whose restriction to X contains ξ as a constituent. Here the normal subgroup X is determined by the character ξ.

There is some notation specific to Dade's Conjecture which will be explained in Section 1.4. In addition, the following notation will be used throughout:

$\mathbf{M_c}, \mathbf{M_c}.2$	McLaughlin Simple group, extension of $\mathbf{M_c}$ by $\mathbb{Z}_2$.
Q_n	Generalized quaternion group of order n.
D_n	Dihedral group of order n.

$\mathbf{M}_{10}$	Mathieu group on 10 letters.
p	Almost invariably a prime number.
$\mathrm{GF}(q)$	Finite field with q elements, q a power of p.
$\mathrm{GL}(n,q) = \mathrm{GL}_n(q)$	General Linear Group of dimension n over $\mathrm{GF}(q)$.
$\mathrm{SL}(n,q) = \mathrm{SL}_n(q)$	Special Linear subgroup of $\mathrm{GL}(n,q)$,
	consisting of all matrices of determinant 1.
$\mathfrak{S}(X)$	Group of permutations of the set X.
$\mathfrak{S}_n$	Symmetric group on n letters.
$\mathfrak{A}_n$	Alternating subgroup of $\mathfrak{S}_n$,
	consisting of all permutations of sign $+1$.
$\mathbb{Z}_m, \mathbb{Z}_m^n$	Cyclic group of order m, direct product of n copies of $\mathbb{Z}_m$.
$G, G^{\#}$	Arbitrary finite group, non-trivial elements of that group.
$n.G$	Central extension of a cyclic group of order n by G.
G/N	Quotient group of G by a normal subgroup N.
$N : G = N \rtimes G$	Semidirect product of a group N with G.
$N.G$	Group with normal subgroup N and quotient $G \cong (N.G)/N$.
$B \wr G$	Wreath product of a group B with a permutation group
	G of degree n, giving a group of the form $B^n \rtimes G$.
$\mathrm{Z}(G)$	Center of G.
$\Phi(G)$	Frattini subgroup of G.
$\mathrm{O}_p(G)$	Largest normal p-subgroup of G, also called the *p-core* of G.
$\langle x_1, \cdots, x_n \rangle$	Group generated by x_1, x_2,$\cdots$,x_n.

$\Omega(P)$	Subgroup $\langle x \in P \mid x^p = 1 \rangle$ of P, where P is a p-group.
$\Omega\mathrm{Z}(P)$	$\Omega(\mathrm{Z}(P))$, largest exponent p-subgroup of $\mathrm{Z}(P)$.
$[x_1, x_2]$	$x_1^{-1}x_2^{-1}x_1x_2$, commutator of x_1 with x_2.
H, K	Arbitrary subgroups of G.
$[H, K]$	$\langle [h, k] \mid h \in H, k \in K \rangle$, commutator subgroup of H with K.
$G' = [G, G]$	Derived subgroup of G.
$G_{ab} = G/G'$	Largest abelian quotient of G.
$H^x = x^{-1}Hx$	Conjugate subgroup of H by x.
$x_1^{x_2} = x_2^{-1}x_1x_2$	Conjugate of x_1 by x_2.
$\mathrm{N}_G(X)$	Normalizer in G of a subset X of some G-set,
	where X will often be a subgroup of G.
$\mathrm{C}_G(X)$	Centralizer in G of a subset X of some G-set.
ψ, ψ^G	Complex character of H, character of G induced by ψ.
$\mathrm{I}_G(\psi)$	Stabilizer or *inertial subgroup* of ψ in G.
$\overline{\psi}$	Complex conjugate of ψ.
$\mathsf{B}(\psi)$	p-block of H containing ψ.
b^G	If defined, the p-block of G induced,
	in the sense of Brauer, from the p-block b of H.
$V^* = \mathrm{Hom}_{\mathfrak{F}}(V, \mathfrak{F})$	Dual space of V, where V is a vector space over a field $\mathfrak{F}$.
$(nA), (nB), \cdots$	Atlas notation for conjugacy classes.

Notation used throughout the thesis

1.2. Representation Theory

Block Theory. Let G be a finite group. We let $\mathfrak{R}$ be a local principal ideal domain with field of fractions $\mathfrak{F}$ of characteristic zero, unique maximal ideal $\mathfrak{P}$ equal to its Jacobson Radical $\mathrm{J}(\mathfrak{R})$ and residue field $\mathfrak{K} = \mathfrak{R}/\mathrm{J}(\mathfrak{R})$ of prime characteristic p.

We have an $\mathfrak{R}$-order $\mathfrak{R}G$ which is naturally embedded in the group algebra $\mathfrak{F}G = \mathfrak{F} \bigotimes_{\mathfrak{R}} \mathfrak{R}G$ of G over $\mathfrak{F}$. We assume that $\mathfrak{F}G$ is a split $\mathfrak{F}$-algebra. In these circumstances the center of the $\mathfrak{R}$-order $\mathfrak{R}G$, denoted by $\mathfrak{Z}$, has the property that $\mathfrak{Z}/\mathrm{J}(\mathfrak{Z}) \cong \mathfrak{K} \bigoplus \cdots \bigoplus \mathfrak{K}$ as vector spaces over $\mathfrak{K}$ by [**Da92**, §4].

We denote by $\mathrm{Z}(\mathfrak{F}G)$ the center of the group algebra $\mathfrak{F}G$. For our purposes a *p-block* of G corresponds to a maximal ideal of $\mathfrak{Z}$. Thus each p-block B is associated to an epimorphism $\omega_{\mathsf{B}} : \mathfrak{Z} \to \mathfrak{K}$ of commutative $\mathfrak{R}$-algebras. We denote by M_{B} the maximal ideal of $\mathfrak{Z}$ which is the kernel of ω_{B}. The blocks B arise in the following way. Let χ be an irreducible complex character of G. Then

$$e_\chi = \frac{\chi(1_G)}{|G|} \sum_{g \in G} \chi(g^{-1}) g$$

is the centrally primitive idempotent of $\mathfrak{F}G$ associated to χ. This means that e_χ lies in $\mathrm{Z}(\mathfrak{F}G)$, it is an idempotent, and the ideal $\mathfrak{F}Ge_\chi$ of $\mathfrak{F}G$ is a simple ideal of $\mathfrak{F}G$. From Wedderburn's Theorem and the fact that $\mathfrak{F}G$ is split, $\mathfrak{F}Ge_\chi$ is a complete matrix algebra over $\mathfrak{F}$. Thus if $\zeta \in \mathrm{Z}(\mathfrak{F}G)$, we can find $f \in \mathfrak{F}$ such that $e_\chi \zeta = f e_\chi$. The map sending ζ to f is $\chi/\chi(1)$.

Now $\chi/\chi(1)$ restricts to a map $\mathfrak{Z} \to \mathfrak{F}$, where the image of $\mathfrak{Z}$ is an $\mathfrak{R}$-suborder of $\mathfrak{F}$. Thus the image coincides with $\mathfrak{R}$. We let K_χ denote the kernel in $\mathfrak{Z}$ of this map. So K_χ is a prime ideal of $\mathfrak{Z}$.

LEMMA 1.2.1. $\cap_{\chi \in \mathrm{Irr}(G)} K_\chi = 0$

PROOF. This comes from looking at the corresponding intersection of kernels in $\mathrm{Z}(\mathfrak{F}G)$. □

Since $\cap_{\chi \in \mathrm{Irr}(G)} K_\chi \subset M_{\mathsf{B}}$ and M_{B} is a prime ideal, there exists $\chi \in \mathrm{Irr}(G)$ such that $K_\chi \subset M_{\mathsf{B}}$. But then M_{B} corresponds to a maximal ideal of $\mathfrak{R} = \mathfrak{Z}/K_\chi$. Hence M_{B} equals the inverse image $\mathfrak{P}\mathfrak{Z} + K_\chi$ of $\mathfrak{P}$ in $\mathfrak{Z}$. We say χ *lies in* B if K_χ is contained in M_{B}, or alternatively if the epimorphism ω_{B} factors through the map $\chi/\chi(1)$. The set of irreducible characters of G lying in the block B will be denoted by $\mathrm{Irr}(\mathsf{B})$.

Heller's Theorem, blocks and block idempotents.

THEOREM 1.2.2 (Heller's Theorem). *Suppose $\mathfrak{R}$ is a local PID with maximal ideal $\mathfrak{P} = \pi\mathfrak{R}$, with field of fractions $\mathfrak{F}$ of characteristic zero, and residue field $\mathfrak{R}/\mathfrak{P} = \mathfrak{K}$ of prime characteristic p. If A is a split semisimple $\mathfrak{F}$-algebra and $\mathfrak{O}$ is a full $\mathfrak{R}$-order inside A, then given an idempotent $\overline{e}$ of $\mathfrak{O}/\mathrm{J}(\mathfrak{O})$, there exists an idempotent e of $\mathfrak{O}$ such that $\overline{e} = e + \mathrm{J}(\mathfrak{O})$.*

PROOF. A proof of this well known theorem can be found in [**CR81**, I, 30.18]. □

LEMMA 1.2.3. *If R is any ring with identity, and e and f are idempotents of R such that $e \equiv f$ modulo $\mathrm{J}(R)$ then there exists $j \in \mathrm{J}(R)$ such that $e^{1+j} = f$.*

PROOF. The proof of this lemma is standard and has been omitted. $\square$

In our situation, since $\mathfrak{Z}/\,\mathrm{J}(\mathfrak{Z}) \cong \mathfrak{K} \bigoplus \cdots \bigoplus \mathfrak{K}$, we have that each block B of G corresponds to a primitive (central) idempotent $\overline{e_{\mathsf{B}}}$ of $\mathfrak{Z}/\,\mathrm{J}(\mathfrak{Z})$. By Heller's Theorem 1.2.2, to each such idempotent $\overline{e_{\mathsf{B}}}$ there corresponds an idempotent e_{B} of $\mathfrak{Z}$, such that $e_{\mathsf{B}} + \mathrm{J}(\mathfrak{Z}) = \overline{e_{\mathsf{B}}}$. Since $\mathfrak{Z}$ is commutative, Lemma 1.2.3 shows that such an idempotent is uniquely determined. It is also clearly primitive. Hence the blocks B of G correspond one-to-one to the primitive (central) idempotents of $\mathfrak{Z}$, which in turn correspond one-to-one with the indecomposable $\mathfrak{R}$-order summands of $\mathfrak{R}G$.

LEMMA 1.2.4. *The block* B *of* G *corresponds to the idempotent* e *of* $\mathfrak{Z}$ *if and only if* $\omega_{\mathsf{B}'}(e) = \delta_{\mathsf{B}',\mathsf{B}}$ *for all blocks* B' *of* G*, where* $\delta_{i,j}$ *is zero or one depending on whether* i *and* j *are equal or not.*

PROOF. This is immediate from our definition of a block B. $\square$

LEMMA 1.2.5. $e_{\mathsf{B}} = \sum\limits_{\chi \in \mathrm{Irr}(\mathsf{B})} e_\chi$. *If* $I \subset \mathrm{Irr}(G)$ *and* $\sum\limits_{\chi \in I} e_\chi$ *is in* $\mathfrak{R}G$*, then* I *is the union of* $\mathrm{Irr}(\mathsf{B}(\chi))$*, where* χ *runs over the elements of* I *and* $\mathsf{B}(\chi)$ *is the* p*-block of* G *containing* χ.

PROOF. The first statement is proved in [**CR81**, II, 56.25]. The second statement follows from the discussion above. $\square$

The Defect of a Block. There are many ways to define the family of defect groups of a p-block B of G. If C is a conjugacy class of G then a *defect group* of C is a Sylow p-subgroup of $\mathrm{C}_G(\sigma)$ for any $\sigma \in C$. The family of defect groups of C clearly forms a single G-orbit of p-subgroups of G. We will denote by $\tilde{C}$ the class sum $\sum_{\sigma \in C} \sigma$ of C. So $\tilde{C} \in \mathrm{Z}(\mathfrak{F}G)$. Define

$$\Upsilon(\mathsf{B}) = \big\{P \mid P \text{ is a defect group of some conjugacy class } C \text{ of } G \text{ satisfying } \omega_{\mathsf{B}}(\tilde{C}) \neq 0\big\}. \tag{1.2.6}$$

A *defect group* D of B is any minimal element of $\Upsilon(\mathsf{B})$. It is not immediately obvious, but the family of all defect groups of B forms a single G-orbit, denoted $\mathrm{Def}(\mathsf{B})$, of p-subgroups of G. See [**NT89**, 5.1.11(ii)].

Definition 1.2.7. The *defect* $\mathrm{d}(\mathsf{B})$ of a p-block B of G is a non-negative integer determined by

$$p^{\mathrm{d}(\mathsf{B})} = |D|, \quad \text{where } D \in \mathrm{Def}(\mathsf{B}).$$

Proposition 1.2.8. *Let B be a p-block of G of defect d. Then d is the maximum integer such that*

$$p^d \mid \frac{|G|}{\chi(1)} \quad \textit{for some } \chi \in \mathrm{Irr}(\mathsf{B}).$$

Proof. This follows from [**CR81**, II, 56.33, 56.41]. □

We mention that the p-block B_0 of G containing the trivial character is called the *principal block* of G. Its defect groups are the Sylow p-subgroups of G. At the other extreme, a p-block B of G of defect 0 is known to contain a single ordinary irreducible character χ_{B}, which satisfies $p \nmid |G|/\chi_{\mathsf{B}}(1)$. Moreover, any irreducible character satisfying this equation lies in a p-block of defect 0.

The following lemma will be used repeatedly in this thesis, usually without being explicitly referenced.

LEMMA 1.2.9. *If G has a normal p-subgroup P such that $\mathrm{C}_G(P) \leq P$, then the principal p-block B_0 is the unique p-block of G.*

PROOF. See [**AL93**, Q2, p112] or [**Ft82**, V.3.11]. $\square$

Some Clifford Theory. Clifford theory connects the representations of a finite group to those of a normal subgroup. In the following G will denote a finite group, N a normal subgroup of G and χ an irreducible character of N.

THEOREM 1.2.10. *Suppose G/N is cyclic and χ is invariant in G. Then χ extends to G.*

PROOF. A proof can be found in [**Is76**, 11.22]. $\square$

Next we have a very useful result which says that the extendibility of a character depends only on the Sylow subgroups of the quotient group.

THEOREM 1.2.11. *Suppose χ is invariant in G. Then χ extends to G if and only if it extends to every subgroup $P \geq N$ of G for which P/N is a Sylow subgroup of G/N.*

PROOF. A proof can be found in [**Is76**, 11.31]. $\square$

We can in fact focus attention on those Sylow p-subgroups for which $p \mid |N|$. Let V be an irreducible right G-module, and let $\chi \in \mathrm{Irr}(G)$ be the irreducible character of the corresponding representation $\tau : G \to \mathrm{End}(V)$ of G on V. We give the following two definitions, which can be found in [**Is76**, p 88].

Definition 1.2.12. The *determinant character* $\lambda = \det(\chi)$, of χ is defined by setting $\lambda(g) = \det(T(g))$, where $T(g)$ is the matrix of $\tau(g)$ with respect to some choice of basis for V, and g is any element of G.

Definition 1.2.13. The *determinantal order* $\mathrm{o}(\chi) = \mathrm{o}(\lambda)$ of χ is the order of the determinant character λ as an element of the group of linear characters of G.

Thus $\mathrm{o}(\chi) = |G : \ker(\lambda)|$.

Theorem 1.2.14. *Suppose χ is invariant in G and*

$$\big(|G : N|, \mathrm{o}(\chi)\chi(1)\big) = 1.$$

Then there is a unique extension $\tilde{\chi}$ of χ to G with the property that $\mathrm{o}(\tilde{\chi}) = \mathrm{o}(\chi)$. In particular, χ has such an extension if $\big(|G : N|, |N|\big) = 1$.

Proof. This is a theorem due to Gallagher, and a proof can be found in [**Is76**, 8.16]. $\square$

Generally we will be interested in whether or not χ extends to its stabilizer $\mathrm{I}_G(\chi)$ in G. The following provides the solution when N is abelian and complemented in G.

Theorem 1.2.15. *Suppose $G = N \rtimes K$ and N is abelian. Then every $\chi \in \mathrm{Irr}(N)$ can be extended to $\tilde{\chi} \in \mathrm{Irr}(\mathrm{I}_G(\chi))$. Moreover $\mathrm{I}_G(\chi) = N \rtimes \mathrm{I}_K(\chi)$.*

Proof. A proof can be found in [**CR81**, I, 11.8]. $\square$

The next theorem gives a complete description of what happens when a character of a normal subgroup extends to its stabilizer.

Theorem 1.2.16. *Suppose* χ *extends to a character* $\tilde{\chi}$ *of its stabilizer* $S = \mathrm{I}_G(\chi)$ *in* G*. Then*

(1) $\tilde{\chi} \in \mathrm{Irr}(S)$;

(2) $(\omega\tilde{\chi})^G \in \mathrm{Irr}(G)$ *for each* $\omega \in \mathrm{Irr}(S \mod N)$;

(3) $\chi^G = \sum_{\omega \in \mathrm{Irr}(S \bmod N)} \omega(1)(\omega\tilde{\chi})^G$ *gives* χ^G *as a linear combination of the distinct irreducible characters* $(\omega\tilde{\chi})^G$ *of* G *lying over the* G*-orbit of* χ.

Proof. A proof can be found in [**CR81**, I, 11.5]. □

The following is in fact a special case of this last theorem.

Theorem 1.2.17. *Suppose* $\mathrm{C}_G(x) \leq N$ *for all* $x \in N^\times$*. Then*

(1) *For* $\omega \in \mathrm{Irr}(N)^\times$ *we have* $\mathrm{I}_G(\omega) = N$ *and* $\omega^G \in \mathrm{Irr}(G)$;

(2) *For* $\chi \in \mathrm{Irr}(G)$ *with* $N \not\subseteq \ker(\chi)$, *we have* $\chi = \omega^G$, *for some* $\omega \in \mathrm{Irr}(N)^\times$;

(3) *If* ω_1 *and* $\omega_2 \in \mathrm{Irr}(N)$ *are conjugate by an element of* G, *then* $\omega_1^G = \omega_2^G$. *Hence there is a 1-1 correspondence between the elements of* $\mathrm{Irr}(G \mid N)$ *and the* G*-conjugacy classes of non-trivial characters of* N.

Proof. This theorem is due to Frobenius. A proof can be found in [**CR81**, I, 11.11]. □

A group G possessing a normal subgroup N satisfying the hypothesis of the above theorem is called a *Frobenius Group*. The normal subgroup N is called a *Frobenius Kernel*. It is a fact that in these circumstances $G = N \rtimes K$, where the complement K is referred to as a *Frobenius Complement* to N.

If P is an extra-special group of order p^{1+2n} for p prime and $n \geq 0$, it can be shown easily, using Clifford Theory and results on alternating bilinear forms, that P has characters of the following degrees.

LEMMA 1.2.18. *Suppose P is an extra-special group of order p^{1+2n} with center $Z(P)$ of order p. Then,*

$$\mathrm{Deg}(P) = \{1^{p^{2n}}, (p^n)^{(p-1)}\}.$$

Moreover the $p-1$ irreducible characters of P of degree p^n are zero outside $Z(P)$ and each lies over exactly one irreducible character of $Z(P)$.

PROOF. A proof of this result can be found in [**Hu67**, 16.14]. □

In many cases we shall be dealing with a group G acting as automorphisms of an abelian group Y. Since Clifford theory is concerned with orbits of characters, the following lemma is useful when we are trying to compute the characters of $\mathrm{Y} \rtimes G$ induced from Y.

LEMMA 1.2.19. *If G is a group acting as automorphisms of an abelian group Y, then the number of G-orbits on $\mathrm{Y}^{\#}$ equals the number of G-orbits on $\mathrm{Irr}(\mathrm{Y})^{\#}$.*

PROOF. The permutation characters of G afforded by the G-sets $\mathrm{Y}^{\#}$ and $\mathrm{Irr}(\mathrm{Y})^{\#}$ are the same. The result now follows from [**CR81**, I, Ex. 10.2(ii)]. □

We also have the following:

Lemma 1.2.20. *Let G be a group acting as linear transformations of a finite-dimensional vector space V over a field K, and V^* be the dual KG-module $\mathrm{Hom}_K(V, K)$. Then*

1. $\mathrm{Dim}_K(\mathrm{C}_{V^*}(G)) = \mathrm{Dim}_K(V/[V, G])$*;*
2. *If K has finite prime order p and G is a finite p'-group, then* $\mathrm{Dim}_K(\mathrm{C}_V(G)) = \mathrm{Dim}_K(\mathrm{C}_{V^*}(G))$.

Proof. 1. Let $\psi \in V^*$. Then

$$\begin{aligned} \psi \in \mathrm{C}_{V^*}(G) &\iff \psi(v^g) = \psi(v), \ \forall v \in V, g \in G \\ &\iff \psi(-v + v^g) = 0, \ \forall v \in V, g \in G \\ &\iff \psi([v, g]) = 0, \ \forall v \in V, g \in G \\ &\iff [V, G] \leq \ker(\psi). \end{aligned}$$

So $\mathrm{C}_{V^*}(G)$ is isomorphic to the dual space of $V/[V, G]$. Hence $\mathrm{Dim}_K(\mathrm{C}_{V^*}(G)) = \mathrm{Dim}_K(V/[V, G])$.

2. Since V is abelian, $V = \mathrm{C}_V(G) \times [V, G]$ by [**As86**, 24.6]. So $\mathrm{Dim}_K(V/[V, G]) = \mathrm{Dim}_K \mathrm{C}_V(G)$. The result follows using part 1. □

Finally, we present a result which we will use frequently to compute the character degrees of "small" groups. We say $N \leq G$ is *subnormal* in G if there is a sequence of subgroups of G of the form

$$N \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_k \trianglelefteq G.$$

THEOREM 1.2.21. *Let $N \leq G$ be subnormal and abelian. Then $\chi(1) \mid |G : N|$ for every $\chi \in \mathrm{Irr}(G)$.*

PROOF. This theorem is due to Ito. A proof may be found in [**Is76**, 11.30]. $\square$

1.3. Radical p-Chains

The objects of our investigation will be the radical p-chains of the McLaughlin Group.

Let G be a finite group and p any prime. A *p-chain* C of G is any non-empty, strictly increasing chain

$$C\colon P_0 < P_1 < P_2 < \cdots < P_n$$

of p-subgroups P_i of G. The *length* $|C|$ of the chain is the number n of inclusions. For each $i = 0,1,\cdots,n$, the chain C_i is the *initial subchain*

$$C_i : P_0 < P_1 < P_2 < \cdots < P_i$$

of C. The family of all p-chains of G will be denoted by $\mathcal{P}$. This is a G-set, with G acting by conjugation. If C is the chain above and $g \in G$, then the *conjugate* chain C^g is given by

$$C^g : P_0^g < P_1^g < P_2^g < \cdots < P_n^g.$$

The *stabilizer* of C in G is the set of all elements of G which conjugate C to itself. Hence it is the intersection

$$\mathrm{N}_G(C) = \mathrm{N}_G(P_0) \cap \mathrm{N}_G(P_1) \cap \cdots \cap \mathrm{N}_G(P_n).$$

We let $\mathcal{P}/G$ denote an arbitrary set of representatives of the G-orbits in $\mathcal{P}$.

LEMMA 1.3.1. *If C is a p-chain and* b *is a p-block of* $\mathrm{N}_G(C)$*, then the induced block* b^G *is defined.*

PROOF. See [**KR89**, 3.2]. ☐

We recall that a *radical p-subgroup* of G is a p-subgroup R satisfying

$$R = \mathrm{O}_p\big(\mathrm{N}_G(R)\big).$$

A *radical p-chain* C of G is one satisfying

$$P_0 = \mathrm{O}_p(G), \qquad P_i = \mathrm{O}_p\big(\mathrm{N}_G(C_i)\big), \quad \text{for } i = 1, 2, \cdots, n.$$

Thus P_0 is the p-core of G, and P_i is a radical p-subgroup of the normalizer of the initial subchain C_{i-1} for $i = 1, 2 \cdots, n$. The fact that P_0 is a normal subgroup of G implies that P_1 is a radical p-subgroup of G. In general P_i, for $i > 1$, need not be a radical p-subgoup of G.

We let $\mathcal{R}$ denote the family of all radical p-chains of G. It is clear that $\mathcal{R}$ is closed under conjugation by elements of G. We let $\mathcal{R}/G$ denote an arbitrary set of representatives for the orbits of G on $\mathcal{R}$.

There are other natural families of p-chains of G. The *normal p-chains* are those satisfying $P_i \trianglelefteq P_n$, for $i = 0, 1, \ldots, n$, while the *elementary abelian p-chains* are composed of elementary abelian p-groups. These two families are denoted by $\mathcal{N}$ and $\mathcal{E}$ respectively. We let $\mathcal{P}'$, $\mathcal{N}'$ and $\mathcal{E}'$ denote the subfamilies of all p-chains C in $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{E}$, respectively, such that the first p-subgroup in C is $P_0 = \{1\}$.

The following is very useful when dealing with radical p-subgroups.

THEOREM 1.3.2. *If R is a radical p-subgroup of G, and P is any p-subgroup of G satisfying* $\mathrm{N}_G(R) \leq \mathrm{N}_G(P)$*, then* $P \leq R$.

PROOF. This proof is taken from [**Da92**, 1.3]. Since R normalizes P, the product PR is a p-subgroup of G. Let $x \in \mathrm{N}_G(R)$. Then x also normalizes P. So x normalizes RP. Hence $\mathrm{N}_G(R) \leq \mathrm{N}_G(PR)$. If $y \in \mathrm{N}_{PR}(R)$, then $y^x \in \mathrm{N}_{PR}(R)$ also. Hence $\mathrm{N}_G(R) \leq \mathrm{N}_G\big(\mathrm{N}_{PR}(R)\big)$. In particular, $\mathrm{N}_{PR}(R)$ is a normal p-subgroup of $\mathrm{N}_G(R)$ containing R. This forces $R = \mathrm{N}_{PR}(R)$. So $PR = R$ since PR is nilpotent. Hence $P \leq R$. □

Interesting consequences of Theorem 1.3.2 include the following.

COROLLARY 1.3.3. $\mathrm{O}_p(G)$ *is the unique minimal radical p-subgroup of G under inclusion.*

PROOF. If R is a radical p-subgroup of G, then $\mathrm{N}_G(R)$ normalizes $\mathrm{O}_p(G)$. □

COROLLARY 1.3.4. *Every radical p-subgroup R of G is the intersection of the family of all Sylow p-subgroups of G containing it.*

PROOF. It is enough to note that the intersection mentioned above contains R and is normalized by $\mathrm{N}_G(R)$. □

Radical p-subgroups behave extremely well with respect to normal subgroups and factor groups.

PROPOSITION 1.3.5. *Suppose $\eta : G^* \to G$ is an epimorphism of finite groups whose kernel K is the direct product $K = Q \times Z$ of a normal p-subgroup Q of G with a* *central p'-subgroup Z of G. Then if P is a p-subgroup of G, there is a unique p-subgroup $P^* = \hat{\eta}(P)$ of G^* containing Q such that $\eta(P^*) = P$. The resulting map $\hat{\eta}$ is an inclusion-preserving bijection of all p-subgroups of G onto all p-subgroups of G^* containing Q. Moreover $\hat{\eta}$ sends the radical p-subgroups of G one-to-one onto the radical p-subgroups of G^*. We can define $\hat{\eta}(C)$, for any p-chain C of G^*, as the p-chain of G formed by the images of the constituent p-subgroups of C under $\hat{\eta}$. Then $\hat{\eta}$ sends the radical p-chains of G one-to-one onto the radical p-chains of G^*.*

PROOF. This follows from Propositions 2.2 and 2.7 of [**Da94**]. $\square$

PROPOSITION 1.3.6. *Say G is a finite group and $N \trianglelefteq G$. Then*

(1) If R is a radical p-subgroup of G then $P = R \cap N$ is a radical p-subgroup of N with $\mathrm{N}_G(P) \geq \mathrm{N}_G(R)$ and $\mathrm{O}_p\big(\mathrm{N}_G(P)\big) \leq R$;

(2) If P is a radical p-subgroup of N then $\mathrm{O}_p\big(\mathrm{N}_G(P)\big)$ is a radical p-subgroup of G and $P = \mathrm{O}_p\big(\mathrm{N}_G(P)\big) \cap N$.

PROOF. (1) Suppose R is radical in G and $P = R \cap N$. Then $\mathrm{N}_G(P) \geq \mathrm{N}_G(R)$ since $N \trianglelefteq G$. Set $Q = \mathrm{O}_p\big(\mathrm{N}_N(P)\big)$. Then Q is characteristic in $\mathrm{N}_N(P)$. So $\mathrm{N}_G(Q) \geq \mathrm{N}_G\big(\mathrm{N}_N(P)\big) \geq \mathrm{N}_G(R)$. By Theorem 1.3.2 we have $Q \leq R$. But then $Q \leq R \cap N = P$. So $Q = P$ and P is radical in N. Since $\mathrm{N}_G(P) \geq \mathrm{N}_G(R)$, it follows from Theorem 1.3.2 that $\mathrm{O}_p\big(\mathrm{N}_G(P)\big) \leq R$.

(2) Suppose P is radical in N. Then $\mathrm{O}_p\big(\mathrm{N}_G(P)\big) \cap N$ is a normal p-subgroup of $\mathrm{N}_N(P)$ containing P. So $P = \mathrm{O}_p\big(\mathrm{N}_G(P)\big) \cap N$. Since $N \trianglelefteq G$, it follows that $\mathrm{N}_G\big(\mathrm{O}_p\big(\mathrm{N}_G(P)\big)\big) \leq \mathrm{N}_G(P) \leq \mathrm{N}_G\big(\mathrm{O}_p\big(\mathrm{N}_G(P)\big)\big)$. Hence $\mathrm{O}_p\big(\mathrm{N}_G(P)\big)$ is radical in G. $\square$

Let P be a p-subgroup of G. Then $P \triangleleft \mathrm{N}_G(P)$. So $P \leq \mathrm{O}_p\big(\mathrm{N}_G(P)\big)$. Moreover $\mathrm{N}_G(P) \leq \mathrm{N}_G\left(\mathrm{O}_p\big(\mathrm{N}_G(P)\big)\right)$. Let $P_1 = \mathrm{O}_p\big(\mathrm{N}_G(P)\big)$, and in general let $P_{n+1} = \mathrm{O}_p\big(\mathrm{N}_G(P_n)\big)$, for $n = 1, 2, \cdots$. Then the P_n's form a non-decreasing chain of subgroups of G. So for some $N \geq 1$ we have $P_{N+1} = P_N$. Then P_N is a radical p-subgroup of G, with $P \leq P_N$ and $\mathrm{N}_G(P) \leq \mathrm{N}_G(P_N)$. This radical p-subgroup is called the *radical closure* of P in G. This argument shows that $\big\{\mathrm{N}_G(R) \mid R \text{ is a radical } p\text{-subgroup of } G\big\}$ contains the set of maximal p-local subgroups of G.

Remark 1.3.7. If R is a p-subgroup of G with center $\mathrm{Z}(R)$, then R is a radical p-subgroup of G if and only if it is a radical p-subgroup of $\mathrm{N}_G\big(\Omega\mathrm{Z}(R)\big)$. Thus we may limit our search for radical p-subgroups of G to radical p-subgroups of the normalizers of elementary abelian p-subgroups of G.

1.4. The Conjectures

The Ordinary Conjecture. Let G be an arbitrary finite group, B a p-block of G and d an integer ≥ 0.

Definition 1.4.1. For any p-chain C of G we denote by $\mathrm{k}(C, \mathsf{B}, d)$ the number of characters $\psi \in \mathrm{Irr}\big(\mathrm{N}_G(C)\big)$ satisfying

$$\mathrm{d}(\psi) = d \quad \text{and} \quad \mathsf{B}(\psi)^G = \mathsf{B}.$$

The integer $\mathrm{k}(C, \mathsf{B}, d)$ depends only on the G-conjugacy class of the p-chain C.

Conjecture 1.4.2. *If* $\mathrm{O}_p(G) = \{1\}$, *and* $\mathrm{d}(\mathsf{B}) > 0$ *in the above situation, then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}(C, \mathsf{B}, d) = 0.$$

We say that this conjecture "holds for G" if it holds for all choices of blocks B and non-negative integers d.

Knörr and Robinson have shown in [**KR89**] that the $\mathcal{R}/G$ in the above sum can be replaced by $\mathcal{P}'/G$, $\mathcal{N}'/G$ or $\mathcal{E}'/G$, without changing the value of that sum.

The Invariant Conjecture. We now embed G as a normal subgroup of another finite group E. We fix an epimorphism $\epsilon : E \twoheadrightarrow \overline{E}$ of finite groups with kernel G. So we have an exact sequence

$$1 \to G \xrightarrow{\trianglelefteq} E \xrightarrow{\epsilon} \overline{E} \to 1$$

of finite groups and their homomorphisms. For any p-chain C of G we define $\mathrm{N}_{\overline{E}}(C)$ to be $\epsilon\big(\mathrm{N}_E(C)\big)$. Since the kernel of the epimorphism $\epsilon : \mathrm{N}_E(C) \to \mathrm{N}_{\overline{E}}(C)$ is the normal subgroup $\mathrm{N}_G(C)$ of $\mathrm{N}_E(C)$, we have an exact sequence

$$1 \to \mathrm{N}_G(C) \xrightarrow{\trianglelefteq} \mathrm{N}_E(C) \xrightarrow{\epsilon} \mathrm{N}_{\overline{E}}(C) \to 1$$

of finite groups associated with C.

The above group $\mathrm{N}_E(C)$ acts by conjugation on the set $\mathrm{Irr}\big(\mathrm{N}_G(C)\big)$ of all ordinary irreducible characters ψ of its normal subgroup $\mathrm{N}_G(C)$. We denote by $\mathrm{N}_E(C, \psi)$ the stabilizer in $\mathrm{N}_E(C)$ of any such ψ, and by

$$\mathrm{N}_{\overline{E}}(C, \psi) = \epsilon\big(\mathrm{N}_E(C, \psi)\big)$$

the image of that stabilizer in $\overline{E}$. Since $\mathrm{N}_G(C)$ is contained in $\mathrm{N}_E(C,\psi)$, we have an exact sequence

$$1 \to \mathrm{N}_G(C) \xrightarrow{\trianglelefteq} \mathrm{N}_E(C,\psi) \xrightarrow{\epsilon} \mathrm{N}_{\overline{E}}(C,\psi) \to 1$$

associated with any p-chain C of G and any character $\psi \in \mathrm{Irr}\big(\mathrm{N}_G(C)\big)$.

From now on we fix a p-block B of G, an integer $d \geq 0$, and a subgroup $\overline{F}$ of $\overline{E}$.

Definition 1.4.3. For any p-chain C of G, we let $\mathrm{k}(C, \mathsf{B}, d, \overline{F})$ denote the number of characters $\psi \in \mathrm{Irr}\big(\mathrm{N}_G(C)\big)$ satisfying

$$\mathrm{d}(\psi) = d, \quad \mathsf{b}(\psi)^G = \mathsf{B} \quad \text{and} \quad \mathrm{N}_{\overline{E}}(C,\psi) = \overline{F}.$$

The integer $\mathrm{k}(C, \mathsf{B}, d, \overline{F})$ depends only on the G-conjugacy class of the p-chain C. So the sum in the following conjecture is well defined.

Conjecture 1.4.4. *If* $\mathrm{O}_p(G) = \{1\}$, *and* $\mathrm{d}(\mathsf{B}) > 0$ *in the above situation, then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}(C, \mathsf{B}, d, \overline{F}) = 0.$$

We say that this conjecture "holds for G" if it holds for all choices of groups $\overline{F}$, blocks B and non-negative integers d.

Note that Conjecture 1.4.2 follows by summing the above equation over all the subgroups $\overline{F}$ of $\overline{E}$.

The Projective Conjecture. Next we go back to an arbitrary finite group G. We pick some central extension $Z.G$ of a cyclic group Z by G. So we fix an exact sequence

$$1 \to Z \xrightarrow{\trianglelefteq} Z.G \xrightarrow{\eta} G \to 1$$

of groups such that Z is a cyclic central subgroup of $Z.G$. Then any p-chain C of G has a normalizer $\mathrm{N}_{Z.G}(C)$ in $Z.G$, the inverse image $\eta^{-1}\big(\mathrm{N}_G(C)\big)$ of its normalizer in G.

We fix a faithful linear character ρ of Z in addition to the non-negative integer d. The earlier p-block B of G is now replaced by a p-block B^* of $Z.G$ lying over the block $\mathsf{B}(\rho)$ of Z containing ρ.

Definition 1.4.5. For any p-chain C of G we define $\mathrm{k}(C, \mathsf{B}^*, d \mid \rho)$ to be the number of characters $\psi \in \mathrm{Irr}\big(\mathrm{N}_{Z.G}(C) \mid \rho\big)$ such that $\mathrm{d}(\psi) = d$ and $\mathsf{B}(\psi)$ induces the p-block B^* of $Z.G$.

Note that $\mathrm{k}(C, \mathsf{B}^*, d \mid \rho)$ depends only on the G-conjugacy class of the p-chain C. So the sum in the following conjecture is well defined.

Conjecture 1.4.6. *If in the above situation,* $\mathrm{O}_p(G) = \{1\}$*, and the Sylow p-subgroup Z_p of Z is not a defect group of* B^**, then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\, \mathrm{k}(C, \mathsf{B}^*, d \mid \rho) = 0.$$

We say that this conjecture "holds for $Z.G$" if it holds for all choices of the above characters ρ, blocks B^*, and non-negative integers d.

Chapter 2

The Prime $p = 5$

2.1. The Radical 5-chains of $\mathrm{M_c}$

From [**Con85**] a Sylow 5-subgroup $\mathbf{P}$ of $\mathbf{M_c}$ is of type 5_+^{1+2}, which indicates that it is extra-special of order 5^3 and exponent 5. In particular $\mathrm{Z}(\mathbf{P})$ is cyclic of order 5. Also from the Atlas $\mathrm{N}_{\mathbf{M_c}}(\mathbf{P}) = \mathbf{P} \rtimes \mathbf{D}$ where $\mathbf{D} \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_8$. In particular $|\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})| = 4 \times 750$.

Proposition 2.1.1. *$\mathbf{P}$ is a trivial intersection subgroup of $\mathbf{M_c}$. The elements of $\mathbf{P} \setminus \mathrm{Z}(\mathbf{P})$ form a single $\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$-orbit.*

Proof. Let γ be an element of $\mathrm{Z}(\mathbf{P})^{\#}$. So the order $|\mathrm{C}_{\mathbf{M_c}}(\gamma)|$ of its centralizer is divisible by $|\mathbf{P}| = 125$. From the Atlas the centralizers of elements in $(5A)$ and $(5B)$ have order 750 and 25, respectively. Hence γ must be a member of the class $(5A)$. Since $\mathrm{Z}(\mathbf{P}) = \langle \gamma \rangle$ is normal in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$, the subgroup $\mathrm{C}_{\mathbf{M_c}}(\gamma) \cap \mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$ has index dividing $|\mathrm{Aut}(\langle \gamma \rangle)| = 4$ in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$. So this subgroup has order divisible by $|\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})|/4 = 750$. But $|\mathrm{C}_{\mathbf{M_c}}(\gamma)| = 750$. Thus $\mathrm{C}_{\mathbf{M_c}}(\gamma)$ equals its intersection with $\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$, and therefore is a subgroup of index 4 in that group. Hence $\mathrm{N}_{\mathbf{M_c}}(\mathbf{P}) / \mathrm{C}_{\mathbf{M_c}}(\gamma) \cong \mathrm{Aut}(\langle \gamma \rangle)$. We conclude that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$ is all of $\mathrm{N}_{\mathbf{M_c}}(\langle \gamma \rangle)$.

Choose $\delta \in \mathbf{P}$ from the class $(5B)$ of $\mathbf{M_c}$. From the Atlas $|\mathrm{C}_{\mathbf{M_c}}(\delta)|$ is of order 25. Thus $\mathrm{C}_{\mathbf{M_c}}(\delta)$ is the maximal elementary abelian subgroup of $\mathbf{P}$ containing δ. As

$|\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})| = 25\times 120$, there are 120 elements in the $\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$-conjugacy class of δ. These conjugates must be all the elements of $\mathbf{P}\setminus \mathrm{Z}(\mathbf{P})$. This proves the second statement.

Suppose $\delta \in \mathbf{P}^\eta$ for some $\eta \in \mathbf{M_c}$. Then $\mathrm{C}_{\mathbf{M_c}}(\delta)$ is a maximal elementary abelian subgroup of both $\mathbf{P}$ and $\mathbf{P}^\eta$. Hence $\mathrm{Z}(\mathbf{P}^\eta) \leq \mathbf{P}$. This implies that $\mathrm{Z}(\mathbf{P}^\eta) = \mathrm{Z}(\mathbf{P})$, since each element of $\mathrm{Z}(\mathbf{P}^\eta)^\#$ lies in $(5A)$, and all elements of $\mathbf{P}\setminus \mathrm{Z}(\mathbf{P})$ lie in $(5B)$ by the previous paragraph. But $\mathbf{P}$ is the unique Sylow p-subgroup of $\mathrm{N}_{\mathbf{M_c}}\big(\mathrm{Z}(\mathbf{P})\big)$. So $\mathbf{P}^\eta = \mathbf{P}$. Hence $\mathbf{P}$ is a trivial intersection subgroup of $\mathbf{M_c}$. □

COROLLARY 2.1.2. $\mathbf{D}$ *acts faithfully on* $\mathbf{P}$.

PROOF. If $\delta \in \mathbf{P}\setminus \mathrm{Z}(\mathbf{P})$, then $\mathrm{C}_{\mathbf{M_c}}(\delta) \leq \mathbf{P}$. The result follows. □

From Corollary 1.3.4 any radical p-subgroup of $\mathbf{M_c}$ is the intersection of the Sylow p-subgroups containing it. But $\mathbf{P}$ is a trivial intersection subgroup of $\mathbf{M_c}$. Hence a radical 5-subgroup of $\mathbf{M_c}$ is either trivial or a conjugate of $\mathbf{P}$. It follows that there are exactly two conjugacy classes of radical 5-chains in $\mathbf{M_c}$.

Chain C	Chain Description	$\mathrm{N}_{\mathbf{M_c}}(C)$	$\mathrm{N}_{\mathbf{M_c}.2}(C)$	Parity
C_1	$\{1\}$	$\mathbf{M_c}$	$\mathbf{M_c}.2$	$+$
C_2	$\{1\} < \mathbf{P}$	$\mathbf{P} \rtimes \mathbf{D}$	$\mathbf{P} \rtimes \mathbf{N}$	$-$

TABLE 2.1. The Radical 5-chains of $\mathbf{M_c}$

The description of $\mathrm{N}_{\mathbf{M_c}.2}(C_2)$ comes from the following section.

2.2. Determining the structure of $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{P})$

From the Atlas $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{P}) = \mathbf{P} \rtimes \mathbf{N}$, where we may assume $\mathbf{D} \le \mathbf{N}$, and $|\mathbf{N} : \mathbf{D}| = 2$. We let $\overline{\mathbf{P}}$ denote the factor group $\mathbf{P}/\,\mathrm{Z}(\mathbf{P})$.

Suppose $\eta \in \mathbf{N}\backslash\mathbf{D}$ centralizes $\overline{\mathbf{P}} = \mathbf{P}/\,\Phi(\mathbf{P})$. From [**Go80**, 5.3.2] this implies that η centralizes $\mathbf{P}$. Then by Corollary 2.1.2 we have $\eta^2 = 1$. Hence η lies in the conjugacy class $(2B)$ of $\mathbf{M_c}.2$. But from the Atlas $|\mathrm{C}_{\mathbf{M_c}}(\eta)| = 7920$, which is not divisible by 5^3. This contradicts the fact that η centralizes $\mathbf{P}$. Hence $\mathbf{N}$ acts faithfully on $\overline{\mathbf{P}}$. So we may regard $\mathbf{N}$ as a subgroup of $\mathrm{GL}(2,5) \cong \mathrm{Aut}(\overline{\mathbf{P}})$.

For the rest of this chapter we fix a generator x for the subgroup $\mathrm{X} = \mathrm{O}_3(\mathbf{D}) \cong \mathbb{Z}_3$. Then $\mathbf{N} \le \mathrm{N}_{\mathrm{GL}(2,5)}(\mathrm{X})$. Since $3 \not|\, |\mathrm{GF}(5)^{\#}| = 4$, the centralizer $\mathrm{C}_{\mathrm{GL}(2,5)}(x)$ of x in $\mathrm{GL}(2,5)$ is isomorphic to $\mathrm{GF}(25)^{\#}$. Moreover the normalizer $\mathrm{N}_{\mathrm{GL}(2,5)}(\mathrm{X})$ of X in $\mathrm{GL}(2,5)$ is isomorphic to the semi-direct product $\mathrm{GF}(25)^{\#} \rtimes \mathrm{Gal}\big(\mathrm{GF}(25)\big)$ of $\mathrm{GF}(25)^{\#}$ with the group $\mathrm{Gal}\big(\mathrm{GF}(25)\big) \cong \mathbb{Z}_2$ of all automorphisms of the field $\mathrm{GF}(25)$. Since both $\mathrm{N}_{\mathrm{GL}(2,5)}(\mathrm{X})$ and $\mathbf{N}$ have order 48, it follows that the two groups are identical.

Let

$$\upsilon = \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix} \qquad \in \mathrm{GL}(2,5),$$

and $\Upsilon = \langle \upsilon \rangle$. Then υ has order 24. We may suppose $x = \upsilon^8$. Thus

$$x = \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix}^8 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad \in \mathrm{GL}(2,5).$$

It follows from the discussion above that $\Upsilon = \mathrm{C}_{\mathrm{GL}(2,5)}(x)$.

If we set

$$\iota = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad \in \mathrm{GL}(2,5),$$

and $\mathrm{T} = \langle \iota \rangle$, then ι is an involution, and

$$\upsilon^{\iota} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix} = \upsilon^5.$$

Thus $\Upsilon \rtimes \mathrm{T} \cong \mathrm{GF}(25)^{\#} \rtimes \mathrm{Gal}\big(\mathrm{GF}(25)\big)$. Moreover T normalizes X, so $\mathbf{N} = \Upsilon \rtimes \mathrm{T}$. Let $y = \upsilon^3$ and $\mathrm{Y} = \langle y \rangle$. So $\mathrm{Y} \cong \mathbb{Z}_8$ and

$$y = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix},$$

Also $\Upsilon = \mathrm{X} \times \mathrm{Y}$. Now $\mathbf{D} = \mathrm{X} \rtimes \mathrm{Z}$ for some cyclic group Z of order 8. The 2-group Z is X-conjugate to a subgroup of the Sylow 2-subgroup $\mathrm{Y} \rtimes \mathrm{T}$ of $\mathbf{N}$. Thus we may assume that $\mathrm{Z} \leq \mathrm{Y} \rtimes \mathrm{T}$. As Z is maximal in $\mathrm{Y} \rtimes \mathrm{T}$, it contains $(\mathrm{Y} \rtimes \mathrm{T})' = \langle y^4 \rangle$. The group $\mathbf{D}$ is non-abelian, as $\mathbf{M_c}$ has no elements of order 24. Thus Z corresponds to a maximal subgroup $\overline{\mathrm{Z}}$ of $(\mathrm{Y} \rtimes \mathrm{T})_{ab} \cong \mathbb{Z}_4 \times \mathbb{Z}_2$, such that $\overline{\mathrm{Z}}$ is cyclic, but not contained in $\mathrm{C}_{\mathbf{N}}(\mathrm{X})/\langle y^4 \rangle$. Let $\overline{y}$ and $\overline{\iota}$ denote the images of y and ι, respectively, in $(\mathrm{Y} \rtimes \mathrm{T})_{ab}$. Then $\langle \overline{y}\,\overline{\iota} \rangle$ is the only possibility for $\overline{\mathrm{Z}}$. Thus $\mathrm{Z} = \langle y\iota \rangle$. For the rest of this chapter we let z denote the generator $y\iota$ of Z. So

$$z = y\iota = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}.$$

2.3. The Character Degrees of $\mathbf{P} \rtimes \mathbf{D}$

Since $\mathbf{D}$ is non-abelian but $\mathbf{D}/\mathrm{X}$ is abelian, $\mathbf{D}' = \mathrm{X}$. So $\mathbf{D}$ has 8 linear characters. The subgroup $\langle xz^2 \rangle$ of $\mathbf{D}$ is abelian and of index 2 in $\mathbf{D}$. Hence by Theorem 1.2.21, all character degrees of $\mathbf{D}$ divide 2. We conclude that

$$\mathrm{Deg}(\mathbf{D}) = \{1^8, 2^4\}, \qquad \mathrm{Def}_5(\mathbf{P} \rtimes \mathbf{D} \bmod \mathbf{P}) = \{3^{12}\}. \tag{2.3.1}$$

From Proposition 2.1.1, the elements of $\mathbf{P} \setminus \mathrm{Z}(\mathbf{P})$ form a single $\mathbf{P} \rtimes \mathbf{D}$ conjugacy class. Since $\overline{\mathbf{P}} = \mathbf{P}/\mathrm{Z}(\mathbf{P})$ is abelian, $\overline{\mathbf{P}}^{\#}$ forms a single orbit of $\mathbf{D}$. As $|\overline{\mathbf{P}}^{\#}| = 24 = |\mathbf{D}|$, the group $\overline{\mathbf{P}} \rtimes \mathbf{D}$ is Frobenius. By Theorem 1.2.17 it has a unique character of degree 24 induced from any non-trivial linear character of $\overline{\mathbf{P}}$. Thus

$$\mathrm{Deg}\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathbf{P}/\mathrm{Z}(\mathbf{P})\big) = \{24\}, \qquad \mathrm{Def}_5\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathbf{P}/\mathrm{Z}(\mathbf{P})\big) = \{3^1\}. \tag{2.3.2}$$

We describe how $\mathrm{GL}(2,5)$ acts on $\mathrm{Z}(\mathbf{P})$. Let ν_1, ν_2 be elements of $\mathbf{P}$ whose images form a basis of $\overline{\mathbf{P}}$. If

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix} \in \mathrm{GL}(2,5),$$

then $\nu_1^M \equiv \nu_1^{m_{1,1}} \nu_2^{m_{1,2}}$ and $\nu_2^M \equiv \nu_1^{m_{2,1}} \nu_2^{m_{2,2}}$ modulo $\mathrm{Z}(\mathbf{P})$. Since $\mathbf{P}$ is extra-special, commutation in $\mathbf{P}$ gives a non-zero alternating bilinear form $\overline{\mathbf{P}} \times \overline{\mathbf{P}} \to \mathrm{Z}(\mathbf{P})$. Hence

$[\nu_1, \nu_2]$ generates $\mathrm{Z}(\mathbf{P})$, and we compute

$$\begin{aligned}[][\nu_1, \nu_2]^M = [\nu_1^M, \nu_2^M] &= [\nu_1^{m_{1,1}} \nu_2^{m_{1,2}}, \nu_1^{m_{2,1}} \nu_2^{m_{2,2}}] \\ &= [\nu_1, \nu_2]^{m_{1,1}m_{2,2}} [\nu_2, \nu_1]^{m_{1,2}m_{2,2}} \\ &= [\nu_1, \nu_2]^{m_{1,1}m_{2,2} - m_{1,2}m_{2,2}} \\ &= [\nu_1, \nu_2]^{\det(M)}.\end{aligned}$$

So any element M of $\mathrm{GL}(2,5)$ raises a generator of $\mathrm{Z}(\mathbf{P})$ to its $\det(M)^{\text{th}}$ power.

Since $\det(x) = 1$ and $\det(z) = 3$, the element x centralizes $\mathrm{Z}(\mathbf{P})$, while z sends every element of $\mathrm{Z}(\mathbf{P})$ to its third power. In particular $\mathbf{D}$ is transitive on $\mathrm{Z}(\mathbf{P})^{\#}$. Thus $\mathrm{C}_{\mathbf{D}}\big(\mathrm{Z}(\mathbf{P})\big)$ has index 4 in $\mathbf{D}$. But $\langle xz^4 \rangle \cong \mathbb{Z}_6$ centralizes $\mathrm{Z}(\mathbf{P})$, and is of index 4 in $\mathbf{D}$. We conclude that

$$\langle xz^4 \rangle = \mathrm{C}_{\mathbf{D}}\big(\mathrm{Z}(\mathbf{P})\big). \tag{2.3.3}$$

By Theorem 1.2.18 the extra-special group $\mathbf{P}$ has 4 irreducible characters of degree 5. Let ψ be one such. Then $\psi|_{P \setminus \mathrm{Z}(\mathbf{P})} \equiv 0$. Hence $\mathrm{I}_{\mathbf{D}}(\psi) = \mathrm{C}_{\mathbf{D}}\big(\mathrm{Z}(\mathbf{P})\big)$.

By Theorem 1.2.10 the character ψ extends to $\tilde{\psi} \in \mathrm{Irr}\big(\mathbf{P} \rtimes \langle xz^4 \rangle\big)$. Then by Theorem 1.2.16 the characters of $\mathbf{P} \rtimes \mathbf{D}$ lying over the orbit of ψ are the 6 elements of the set $\big\{(\omega\tilde{\psi})^{\mathbf{P} \rtimes \mathbf{D}} \mid \omega \in \mathrm{Irr}(\langle xz^4 \rangle)\big\}$. Each such character is of degree $20 = 5 \cdot 4$. Thus

$$\mathrm{Deg}\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathrm{Z}(\mathbf{P})\big) = \{20^6\}, \qquad \mathrm{Def}_5\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathrm{Z}(\mathbf{P})\big) = \{2^6\}. \tag{2.3.4}$$

From (2.3.1), (2.3.2), and (2.3.4) we have

$$\mathrm{Deg}(\mathbf{P} \rtimes \mathbf{D}) = \{1^8, 2^4, 24, 20^6\}, \qquad \text{and} \qquad \mathrm{Def}_5(\mathbf{P} \rtimes \mathbf{D}) = \{3^{13}, 2^6\}. \tag{2.3.5}$$

We can now prove the following

PROPOSITION 2.3.6. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_2) = \mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$ *has a unique* 5*-block, which necessarily induces the principal* 5*-block,* B_0*, of* $\mathbf{M_c}$*. Hence*

$$\begin{aligned} &\mathrm{k}(C_2, \mathsf{B}_0, 3) = 13, \qquad \mathrm{k}(C_2, \mathsf{B}_0, 2) = 6, \qquad \textit{and} \\ &\mathrm{k}(C_2, \mathsf{B}_0, d) = 0, \qquad \textit{for all other values of } d. \end{aligned} \tag{2.3.7}$$

PROOF. Since $\mathrm{C}_{\mathbf{P} \rtimes \mathbf{D}}(\mathbf{P}) = \mathrm{Z}(\mathbf{P})$, Lemma 1.2.9 implies that $\mathbf{P} \rtimes \mathbf{D}$ has a unique 5-block. We obtain (2.3.7) from (2.3.5). $\square$

2.4. The Ordinary Conjecture for the prime $p = 5$

From [**Con85**] the group $\mathbf{M_c}$ has 19 characters in its principal block B_0, and 5 blocks of defect 0. We list the characters in the principal block and their defects:

Character	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_{10}	χ_{11}
Degree	1	22	231	252	770	770	896	896	3520	3520
5-Defect	3	3	3	3	2	2	3	3	2	2

Character	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	
Degree	4752	5103	5544	8019	8019	9856	9856	10395	10395	
5-Defect	3	3	3	3	3	3	3	2	2	

Thus

$$\begin{aligned} &\mathrm{k}(C_1, \mathsf{B}_0, 3) = 13, \qquad \mathrm{k}(C_1, \mathsf{B}_0, 2) = 6, \qquad \text{and} \\ &\mathrm{k}(C_1, \mathsf{B}_0, d) = 0, \qquad \text{for all other values of } d. \end{aligned} \tag{2.4.1}$$

THEOREM 2.4.2. *The Ordinary Conjecture holds for the McLaughlin simple group for the prime $p = 5$.*

PROOF. From Conjecture (1.4.2) and Table 2.1 on page 25, we need to prove

$$\mathrm{k}(C_1, \mathsf{B}_0, d) = \mathrm{k}(C_2, \mathsf{B}_0, d) \tag{2.4.3}$$

for all values of d.

We obtain the following terms for the equation above for various values of d from (2.3.7) and (2.4.1):

5-Defect	C_1		C_2
3	13	=	13
2	6	=	6

TABLE 2.2. The Ordinary Conjecture for $p = 5$

The terms in the above equation are zero for all other values of d. This proves the theorem. □

2.5. The Invariant Characters of $\mathbf{P} \rtimes \mathbf{D}$

Recall that $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{P}) = \mathbf{P} \rtimes \mathbf{N}$, where $\mathbf{N} = \Upsilon \rtimes \mathrm{T}$. Since Υ has index 2 in $\mathbf{N}$, we conclude from Theorem 1.2.21 that every character degree of $\mathrm{N}_{\mathrm{GL}(2,5)}(\mathrm{X})$ divides 2. Furthermore $[\upsilon, \iota] = \upsilon^4$, so $\mathbf{N}' = \langle \upsilon^4 \rangle$ is of index 8 in $\mathbf{N}$. Thus

$$\mathrm{Deg}(\mathbf{N}) = \{1^8, 2^{10}\} \tag{2.5.1}$$

We have $\mathrm{Deg}(\mathbf{D}) = \{1^8, 2^4\}$, by (2.3.1). So four of the linear characters of $\mathbf{D}$ are invariant in $\mathbf{N}$, while the remaining four fuse to give two characters of degree 2. The four characters of $\mathbf{D}$ of degree 2 must all be invariant in $\mathbf{N}$. Hence

$$\text{(2.5.2)} \qquad \mathrm{Inv}(\mathbf{D}) = \{1^4, 2^4\} \qquad \text{and} \qquad \mathrm{InvDef}_5(\mathbf{P} \rtimes \mathbf{D} \bmod \mathbf{P}) = \{3^8\}.$$

The unique character of degree 24 of $\mathbf{P} \rtimes \mathbf{D}$ is necessarily invariant in $\mathbf{N}$. Thus

$$\text{(2.5.3)} \quad \mathrm{Inv}\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathbf{P}/\,\mathrm{Z}(\mathbf{P})\big) = \{24\} \qquad \text{and} \qquad \mathrm{InvDef}_5\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathbf{P}/\,\mathrm{Z}(\mathbf{P})\big) = \{3\}.$$

Finally we deal with the six characters of $\mathrm{Irr}\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathrm{Z}(\mathbf{P})\big)$. As $\mathbf{N}$ normalises $\mathbf{P}$, it necessarily normalises $\mathrm{Z}(\mathbf{P})$. Then since $\mathbf{N}$ acts transitively on $\mathrm{Z}(\mathbf{P})^{\#}$, it also acts transitively on $\mathrm{Irr}\big(\mathrm{Z}(\mathbf{P})\big)^{\#}$. Moreover, the stabilizer of any non-trivial linear character of $\mathrm{Z}(\mathbf{P})$ is of index four in $\mathbf{N}$.

Recall from (2.3.3) that $\mathrm{C}_{\mathbf{D}}\big(\mathrm{Z}(\mathbf{P})\big) = \langle xz^4 \rangle$ is cyclic of order six. Since $\det(z^2) = -1$ and $\det(\iota) = -1$, the elements z^2 and ι invert $\mathrm{Z}(\mathbf{P})$. So ιz^2 centralizes $\mathrm{Z}(\mathbf{P})$. Hence $\mathrm{C}_{\mathbf{N}}\big(\mathrm{Z}(\mathbf{P})\big) = \langle xz^4, \iota z^2 \rangle$.

Now

$$\text{(2.5.4)} \qquad (xz^4)^{\iota z^2} = (x^{-1}z^4)^{z^2} = x^{-1}z^4 = (xz^4)^{-1},$$

while

$$\text{(2.5.5)} \qquad (\iota z^2)^2 = z^{10}z^2 = z^{12} = (xz^4)^3.$$

Thus $\mathrm{T}_{12} = \langle xz^4, \iota z^2 \rangle$ is the unique nonabelian group $\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ of order 12 distinct from D_{12} and $\mathfrak{A}_4$. It is readily established that $\mathrm{Deg}(\mathrm{T}_{12}) = \{1^4, 2^2\}$.

Let $\psi \in \mathrm{Irr}\big(\mathbf{P} \mid \mathrm{Z}(\mathbf{P})\big)$. Then $\mathrm{I}_{\mathbf{N}}(\psi) = \mathrm{T}_{12}$. By Theorem 1.2.14 the character ψ extends to a character $\tilde{\psi}$ of $\mathbf{P} \rtimes \mathrm{T}_{12}$. Moreover, by Theorem 1.2.16 the set $\big\{(\omega\tilde{\psi})^{\mathbf{P}\rtimes\mathbf{N}} \mid \omega \in \mathrm{Irr}(\mathrm{T}_{12})\big\}$ gives the irreducible characters of $\mathbf{P} \rtimes \mathbf{N}$ lying over the $\mathbf{P} \rtimes \mathbf{N}$-orbit of ψ, and hence of ζ. So

$$\mathrm{Deg}\big(\mathbf{P} \rtimes \mathbf{N} \mid \mathrm{Z}(\mathbf{P})\big) = \{20^4, 40^2\}. \tag{2.5.6}$$

From (2.3.4) and (2.5.6) we have

$$\mathrm{Inv}\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathrm{Z}(\mathbf{P})\big) = \{20^2\} \quad \text{and} \quad \mathrm{InvDef}_5\big(\mathbf{P} \rtimes \mathbf{D} \mid \mathrm{Z}(\mathbf{P})\big) = \{2^2\} \tag{2.5.7}$$

From (2.5.2), (2.5.3) and (2.5.7) we have

$$\mathrm{Inv}(\mathbf{P} \rtimes \mathbf{D}) = \{1^4, 2^4, 20^2, 24\}, \quad \text{and} \quad \mathrm{InvDef}_5(\mathbf{P} \rtimes \mathbf{D}) = \{3^9, 2^2\} \tag{2.5.8}$$

We can now prove the following

Proposition 2.5.9. *The group $\mathrm{N}_{\mathbf{M_c}}(C_2) = \mathbf{P} \rtimes \mathbf{D}$ has a unique 5-block, which necessarily induces the principal 5-block, B_0, of $\mathbf{M_c}$. Hence*

$$\begin{aligned} &\mathrm{k}(\,C_2, \mathsf{B}_0, 3, \overline{\mathbf{M_c}.2}\,) = 9, \qquad \mathrm{k}(\,C_2, \mathsf{B}_0, 2, \overline{\mathbf{M_c}.2}\,) = 2, \\ &\mathrm{k}(\,C_2, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = 0 \qquad \textit{for all other values of } d. \end{aligned} \tag{2.5.10}$$

Proof. This follows immediately from (2.5.8) and Proposition 2.3.6. □

2.6. The Invariant Conjecture for the prime $p = 5$

From [**Con85**] the block B_0 of $\mathbf{M_c}$ has eleven characters which are invariant in $\mathbf{M_c}.2$. We list these characters and their defects:

Character	χ_1	χ_2	χ_3	χ_4	χ_7	χ_8	χ_{10}	χ_{11}	χ_{13}	χ_{14}	χ_{15}
Degree	1	22	231	252	896	896	3520	3520	4752	5103	5544
5-Defect	3	3	3	3	3	3	2	2	3	3	3

Thus

$$\begin{aligned} &\mathrm{k}(\, C_1, \mathsf{B}_0, 3, \overline{\mathbf{M_c}.2}\,) = 9, \qquad \mathrm{k}(\, C_1, \mathsf{B}_0, 2, \overline{\mathbf{M_c}.2}\,) = 2, \qquad \text{and} \\ &\mathrm{k}(\, C_1, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = 0 \qquad \text{for all other values of } d. \end{aligned} \tag{2.6.1}$$

THEOREM 2.6.2. *The Invariant Conjecture holds for the McLaughlin simple group and the prime $p = 5$.*

PROOF. From Conjecture (1.4.4) and Table 2.1 on page 25, we need to prove

$$\mathrm{k}(\, C_1, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = \mathrm{k}(\, C_2, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,), \tag{2.6.3}$$

for all values of d. We obtain the following terms for the equation above for various values of d from (2.5.10) and (2.6.1):

5-Defect	C_1		C_2
3	9	=	9
2	2	=	2

TABLE 2.3. The Invariant Conjecture for $p = 5$

The terms in the above equation are zero for all other values of d. This proves the theorem. □

REMARK 2.6.4. G. Schneider has also computed the multiset $\mathrm{Deg}(\mathbf{P} \rtimes \mathbf{N})$, using the *CAYLEY* computer program. He used it to prove the weaker Alperin Conjecture for the McLaughlin Group and $p = 5$. For further information see [**BM90**, P465].

2.7. The Projective Conjecture for the prime $p = 5$

We prove in Theorem 3.12.8 that the Schur multiplier $\mathbf{A}$ of $\mathbf{M_c}$ is cyclic of order 3. Hence the only covering groups of $\mathbf{M_c}$ are $\mathbf{M_c}$ and $\widehat{\mathbf{M_c}} \cong 3.\mathbf{M_c}$. By the Atlas there is no central extension of $\mathbf{M_c}.2$. So we need only show that the Projective Conjecture holds for $\widehat{\mathbf{M_c}}$.

THEOREM 2.7.1. *The Projective Conjecture holds for McLaughlin's simple group and the prime $p = 5$.*

PROOF. Let $\rho \in \mathrm{Irr}(\mathbf{A})^{\#}$. Then from the Atlas the group $\widehat{\mathbf{M_c}}$ has two 5-blocks lying over the 5-block of $\mathbf{A}$ containing ρ. One of these blocks has a trivial defect group and contains the single irreducible character χ_{38}; the other block B^* has $\mathbf{P}$ as a defect group and contains the remaining 19 characters of $\mathrm{Irr}(\widehat{\mathbf{M_c}} \mid \rho)$. We list here the elements of $\mathrm{Irr}(\mathsf{B}^* \mid \rho)$:

Character	χ_{25}	χ_{26}	χ_{27}	χ_{28}	χ_{29}	χ_{30}	χ_{31}	χ_{32}	χ_{33}	χ_{34}
Degree	126	126	792	1980	2376	2376	2520	2520	2772	4752
5-Defect	3	3	3	2	3	3	2	2	3	3

Character	χ_{35}	χ_{36}	χ_{37}	χ_{39}	χ_{40}	χ_{41}	χ_{42}	χ_{43}	χ_{44}	
Degree	5103	6336	6336	8019	8019	8064	10395	10395	10395	
5-Defect	3	3	3	3	3	3	2	2	2	

Thus

$$(2.7.2) \qquad \mathrm{k}(C_1, \mathsf{B}^*, 3 \mid \rho) = 13, \qquad \mathrm{k}(C_1, \mathsf{B}^*, 2 \mid \rho) = 6$$

and $\mathrm{k}(C_1, \mathsf{B}^*, d \mid \rho) = 0$ for all other values of d.

The group $\mathbf{A}.\,\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$ has a Sylow 3-subgroup of order 9. From the Atlas character table of $\mathbf{M_c}$, the group $\widehat{\mathbf{M_c}}$ has no elements of order 9. So a Sylow 3-subgroup of $\mathbf{A}.\,\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$ splits over $\mathbf{A}$. Hence the group $\mathbf{A}.\,\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$ itself splits over $\mathbf{A} \cong \mathbb{Z}_3$, using Theorem [**As86**, 10.4]. By (2.3.7) a cohort of characters of $\mathbf{A}.\,\mathrm{N}_{\mathbf{M_c}}(\mathbf{P}) = \mathrm{N}_{\widehat{\mathbf{M_c}}}(C_2)$ lying over ρ contains 19 members. Thirteen of these characters are of defect 3 and six are of defect 2. All of these characters lie in a 5-block of $\mathbf{A}.\,\mathrm{N}_{\mathbf{M_c}}(\mathbf{P})$ inducing the 5-block B^* of $\widehat{\mathbf{M_c}}$. Hence

$$(2.7.3) \qquad \mathrm{k}(C_2, \mathsf{B}^*, 3 \mid \rho) = 13, \qquad \mathrm{k}(C_2, \mathsf{B}^*, 2 \mid \rho) = 6$$

and $\mathrm{k}(C_2, \mathsf{B}^*, d \mid \rho) = 0$, for all other values of d.

From Conjecture (1.4.6) and Table 2.1 on page 25, we need to prove

$$(2.7.4) \qquad \mathrm{k}(C_1, \mathsf{B}^*, d \mid \rho) = \mathrm{k}(C_2, \mathsf{B}^*, d \mid \rho),$$

for all values of d.

From (2.7.2) and (2.7.3), we obtain the following terms for the equation above for various values of d:

5-Defect	C_1		C_2
3	13	$=$	13
2	6	$=$	6

TABLE 2.4. The Projective Conjecture for $p = 5$

The terms in (2.7.4) are zero for all other values of d. This proves the theorem. $\square$

Chapter 3

The Prime p=3

3.1. The Groups $\mathbf{E} \rtimes \mathbf{M}$ and $\mathbf{F} \rtimes \mathbf{L}$

For the remainder of the thesis α will denote a fixed element of the $(3A)$ conjugacy class of $\mathbf{M_c}$. From the Atlas $\mathrm{N}_{\mathbf{M_c}}\big(\langle\alpha\rangle\big) = \mathbf{F} \rtimes \mathbf{L}$, where $\mathbf{F} \cong 3_+^{1+4}$ and $\mathbf{L} \cong 2.\mathfrak{S}_5$. By [**GL83**, p55] a Sylow 2-subgroup of $\mathbf{L}$ is isomorphic to a generalized quaternion group Q_{16} of order 2^4. Thus $\mathbf{L}$ is an isoclinic variant of the group $\mathrm{SL}(2,5) \rtimes \mathbb{Z}_2$. For the remainder of the thesis $\mathbf{C}$ will denote the derived group $\mathbf{L}' \cong \mathrm{SL}(2,5)$ of $\mathbf{L}$. Since $\mathrm{O}_3(\mathbf{L}) = \{1\}$, the group $\mathbf{F}$ is the 3-core of $\mathbf{F} \rtimes \mathbf{L}$. Clearly $\mathrm{Z}(\mathbf{F}) = \langle\alpha\rangle$. Hence

$$\mathbf{F} \rtimes \mathbf{L} = \mathrm{N}_{\mathbf{M_c}}(\mathbf{F}). \tag{3.1.1}$$

LEMMA 3.1.2. *Let $\mathbf{S_L}$ be a Sylow 3-subgroup of $\mathbf{L}$ and let $\mathbf{N_L}$ denote the normalizer of $\mathbf{S_L}$ in $\mathbf{L}$. Then $\mathbf{S_L} \cong \mathbb{Z}_3$ and $\mathbf{N_L} \cong \mathbb{Z}_3 \rtimes Q_8$. Let $\mathbf{Q}$ be a complement to $\mathbf{S_L}$ in $\mathbf{N_L}$. We may choose generators a and b for $\mathbf{Q}$ such that $a \in \mathbf{L}\backslash\mathbf{C}$, $b \in \mathbf{C}$ and both a and b invert $\mathbf{S_L}$.*

PROOF. This is clear from the structure of $\mathbf{L} \cong \mathrm{SL}(2,5).2$. □

We use $\mathbf{W}$ to denote the Sylow 3-subgroup $\mathbf{F} \rtimes \mathbf{S_L}$ of $\mathbf{F} \rtimes \mathbf{L}$. From the Atlas it is also a Sylow 3-subgroup of $\mathbf{M_c}$.

Let τ be a generator of $\mathrm{Z}(\mathbf{Q}) = \mathrm{Z}(\mathbf{L})$. So $a^2 = b^2 = \tau$. From the Atlas the centralizer $\mathbf{H} = \mathrm{C}_{\mathbf{M_c}}(\tau)$ of τ is isomorphic to the universal covering group $2.\mathfrak{A}_8$ of $\mathfrak{A}_8$.

LEMMA 3.1.3. $\mathbf{F} \rtimes \mathbf{C} = \mathrm{C}_{\mathbf{M_c}}(\alpha)$. *Hence* $\mathbf{S_L} \rtimes \langle b \rangle$ *centralizes* α, *while* a *inverts* α.

PROOF. The group $\mathbf{L}$ acts on the group $\langle \alpha \rangle = \mathrm{Z}(\mathbf{F}) \cong \mathbb{Z}_3$. This action induces a homomorphism of $\mathbf{L}$ into the abelian group $\mathrm{Aut}\big(\mathrm{Z}(\mathbf{F})\big) \cong \mathbb{Z}_2$. Hence $\mathbf{C} = \mathbf{L}'$ is contained in the kernel of this homomorphism i.e. it centralizes $\mathrm{Z}(\mathbf{F})$.

Since $\tau \in \mathbf{C}$ centralizes α, the group $\mathrm{Z}(\mathbf{F}) \rtimes \mathbf{L}$ is a subgroup of $\mathbf{H} = \mathrm{C}_{\mathbf{M_c}}(\tau)$. It follows from the structure of $\mathbf{H} \cong 2.\mathfrak{A}_8$ that $\mathrm{Z}(\mathbf{F}) \rtimes \mathbf{L} = \mathrm{N}_{\mathbf{H}}\big(\mathrm{Z}(\mathbf{F})\big)$ and $\mathrm{Z}(\mathbf{F}) \times \mathbf{C} = \mathrm{C}_{\mathbf{H}}(\alpha)$. In particular $\mathbf{L}$ does not centralize α. We conclude that $\mathbf{F} \rtimes \mathbf{C} = \mathrm{C}_{\mathbf{M_c}}(\alpha)$.

Now $\mathbf{S_L} \rtimes \langle b \rangle$ centralizes α because it is a subgroup of $\mathbf{C}$, while $a \in \mathbf{L} \backslash \mathbf{C}$ inverts α because it acts non-trivially on $\mathrm{Z}(\mathbf{F})$. This concludes the lemma. $\square$

We let $\overline{\mathbf{F}}$ denote the quotient group $\mathbf{F}/\,\mathrm{Z}(\mathbf{F})$. So $\overline{\mathbf{F}}$ is elementary abelian of order 3^4. Clearly $\mathbf{L}$ acts on $\overline{\mathbf{F}}$.

PROPOSITION 3.1.4. $\mathrm{C}_{\mathbf{W}}(\tau) = \mathrm{Z}(\mathbf{F}) \times \mathbf{S_L}$. *Hence* τ *inverts* $\overline{\mathbf{F}}$. *Also* $\mathrm{C}_{\mathbf{W}}(\mathbf{Q}) = \{1\}$. *A Sylow* 2*-subgroup of* $\mathbf{F} \rtimes \mathbf{L}$ *acts fixed-point-free on* $\overline{\mathbf{F}}^{\#}$.

PROOF. The group $\mathrm{C}_{\mathbf{W}}(\tau) = \mathbf{W} \cap \mathbf{H}$ is a 3-subgroup of $\mathbf{H} = \mathrm{C}_{\mathbf{M_c}}(\tau)$ containing $\mathrm{Z}(\mathbf{F}) \times \mathbf{S_L}$. From the Atlas a Sylow 3-subgroup of $\mathbf{H} \cong 2.\mathfrak{A}_8$ is elementary abelian of order 3^2. So $\mathrm{C}_{\mathbf{W}}(\tau) = \mathrm{Z}(\mathbf{F}) \times \mathbf{S_L}$. Hence $\mathrm{C}_{\mathbf{F}}(\tau) = \mathrm{Z}(\mathbf{F})$. As τ has order relatively prime to 3, this implies that $\mathrm{C}_{\overline{\mathbf{F}}}(\tau) = \{1\}$. So τ inverts $\overline{\mathbf{F}}$.

Since $a \in \mathbf{Q}$ inverts $\mathrm{C}_{\mathbf{W}}(\tau) = \mathrm{Z}(\mathbf{F}) \times \mathbf{S_L}$, it follows that $\mathrm{C}_{\mathbf{W}}(\mathbf{Q}) = \{1\}$.

Let $\mathbf{Q}_{16}$ be a Sylow 2-subgroup of $\mathbf{F} \rtimes \mathbf{L}$. We may suppose that $\tau \in \mathbf{Q}_{16}$. Since $\mathbf{Q}_{16}$ is generalized quaternion, τ is its unique involution. Hence $\mathbf{Q}_{16}$ acts fixed-point-free on $\overline{\mathbf{F}}$. $\square$

COROLLARY 3.1.5. *The group* $\mathbf{L}$ *acts faithfully on* $\overline{\mathbf{F}}$.

PROOF. This follows immediately from Proposition 3.1.4 and the fact that $\langle \tau \rangle$ is the unique minimal normal subgroup of $\mathbf{L} \cong \mathrm{SL}(2,5).2$. $\square$

LEMMA 3.1.6. $\mathrm{Z}(\mathbf{W}) = \mathrm{Z}(\mathbf{F})$.

PROOF. As $\mathrm{N}_{\mathbf{M_c}}(\mathbf{F}) = \mathbf{F} \rtimes \mathbf{L}$, we have $\mathrm{C}_{\mathbf{M_c}}(\mathbf{F}) \leq \mathbf{F} \rtimes \mathbf{L}$. But $\mathbf{L}$ acts faithfully on $\overline{\mathbf{F}}$, by Corollary 3.1.5. So $\mathrm{C}_{\mathbf{M_c}}(\mathbf{F}) = \mathrm{Z}(\mathbf{F})$. Thus $\mathrm{Z}(\mathbf{W}) \leq \mathrm{Z}(\mathbf{F})$. This implies the lemma since $\mathrm{Z}(\mathbf{F})$ has prime order 3. $\square$

LEMMA 3.1.7. *Suppose* $\mathbf{Q}$ *acts on an elementary abelian group* $\tilde{\mathbf{E}}$ *of order* 3^4, *in such a manner that* $\tau \in \mathbf{Q}$ *inverts* $\tilde{\mathbf{E}}$. *Then any proper non-trivial* $\mathbf{Q}$*-invariant subgroup* $\tilde{\mathbf{F}}$ *of* $\tilde{\mathbf{E}}$ *is a simple* $\mathbf{Q}$*-group of order* 3^2. *Moreover* $\mathbf{Q}$ *acts regularly and transitively on* $\tilde{\mathbf{F}}^{\#}$.

PROOF. Let $\tilde{\mathbf{F}}$ be a proper non-trivial $\mathbf{Q}$-invariant subgroup of $\tilde{\mathbf{E}}$. Then $\mathbf{Q}$ acts faithfully on both $\tilde{\mathbf{F}}$ and $\tilde{\mathbf{E}}/\tilde{\mathbf{F}}$. Hence both subgroups have order $\geq 3^2$. The lemma follows. $\square$

PROPOSITION 3.1.8. $\mathbf{W}'/\,\mathrm{Z}(\mathbf{W}) = [\mathbf{S_L}, \overline{\mathbf{F}}] = \mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S_L})$, *and each of these groups has index* 3^2 *in* $\overline{\mathbf{F}}$ *and order* 3^2. *Let* $\mathbf{E} = \mathbf{W}'\mathbf{S_L}$. *Then* $\mathbf{W}' = \mathbf{E} \cap \mathbf{F}$, *and* $\mathbf{E} = \mathrm{C}_{\mathbf{W}}(\mathbf{W}')$ *is* *elementary abelian of order* 3^4. *Moreover* $[\mathbf{W}, \mathbf{W}'] = \mathrm{Z}(\mathbf{W})$. *So the* 3*-group* $\mathbf{W}$ *is of class three.*

Proof. We have $\mathbf{W}' < \mathbf{F}$, as $\mathbf{F}$ is a maximal subgroup of $\mathbf{W}$. Moreover $\mathrm{Z}(\mathbf{W}) < \mathbf{W}'$, as $\mathbf{W}'$ contains the inverse image of the non-trivial subgroup $[\mathbf{S_L}, \overline{\mathbf{F}}]$ of $\overline{\mathbf{F}}$. Hence $\mathbf{W}'/\,\mathrm{Z}(\mathbf{W})$ is a simple $\mathbf{Q}$-subgroup of $\overline{\mathbf{F}}$ of order 3^2, by Lemma 3.1.7. But $[\mathbf{S_L}, \overline{\mathbf{F}}]$ is a non-trivial $\mathbf{Q}$-subgroup of $\mathbf{W}'/\,\mathrm{Z}(\mathbf{W})$. So $[\mathbf{S_L}, \overline{\mathbf{F}}] = \mathbf{W}'/\,\mathrm{Z}(\mathbf{W})$.

Now $[\mathbf{W}, \mathbf{W}'] \neq \{1\}$, as $\mathrm{Z}(\mathbf{W}) < \mathbf{W}'$. Since $\mathrm{Z}(\mathbf{W})$ is simple, this and [**Rt95**, 5.41] imply that $\mathrm{Z}(\mathbf{W}) \leq [\mathbf{W}, \mathbf{W}']$. Hence $[\mathbf{W}, \mathbf{W}']/\,\mathrm{Z}(\mathbf{W})$ is a proper $\mathbf{Q}$-subgroup of the simple $\mathbf{Q}$-space $\mathbf{W}'/\,\mathrm{Z}(\mathbf{W})$. So $[\mathbf{W}, \mathbf{W}'] = \mathrm{Z}(\mathbf{W})$. We conclude that $\mathbf{W}$ is a 3-group of class three. In particular $\mathbf{W}'$ is abelian.

It follows from the previous paragraph that commutation in $\mathbf{W}$ gives a $\langle\tau\rangle$-invariant bilinear map

$$\frac{\mathbf{W}'}{\mathrm{Z}(\mathbf{W})} \times \mathbf{S_L} \to \mathrm{Z}(\mathbf{W}).$$

But τ inverts $\mathbf{W}'/\,\mathrm{Z}(\mathbf{W})$, while centralizing $\mathbf{S_L}$ and $\mathrm{Z}(\mathbf{W})$. So this map is trivial. Hence $\mathbf{W}'\mathbf{S_L}$ is abelian. Thus $\mathbf{W}'/\,\mathrm{Z}(\mathbf{W}) \leq \mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S_L})$. But $\mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S_L})$ is a non-trivial proper $\mathbf{Q}$-subgroup of $\overline{\mathbf{F}}$. So $\mathbf{W}'/\,\mathrm{Z}(\mathbf{W}) = \mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S_L})$.

Let $\mathbf{E} = \mathbf{W}'\mathbf{S_L}$. Then $\mathbf{W}'$ is a maximal subgroup of $\mathbf{E}$ contained in $\mathbf{F}$. Since $\mathbf{S_L} \cap \mathbf{F} = \{1\}$, we conclude that $\mathbf{E} \cap \mathbf{F} = \mathbf{W}'$. As $\mathbf{W}'\mathbf{S_L}$ is abelian we have

$$\mathbf{E} = \mathrm{C}_{\mathbf{W}'\mathbf{S_L}}(\tau) \times [\mathbf{W}'\mathbf{S_L}, \tau] = \mathrm{Z}(\mathbf{W}) \times \mathbf{S_L} \times [\mathbf{W}'\mathbf{S_L}, \tau],$$

using Proposition 3.1.4 and [**As86**, 24.6]. Since $\mathbf{F}$ has exponent 3, this implies in particular that $\mathbf{E}$ is elementary abelian of order 3^4.

Now $\mathbf{F}$ is a maximal subgroup of $\mathbf{W}$ and $\mathbf{W}' = \mathrm{C}_{\mathbf{F}}(\mathbf{W}')$ is a maximal abelian subgroup of the extra-special group $\mathbf{F}$. So $\mathrm{C}_{\mathbf{W}}(\mathbf{W}') = \mathbf{E}$. $\square$

COROLLARY 3.1.9. *$\mathbf{F}$ is the unique extra-special subgroup of $\mathbf{W}$ of order 3^5.*

PROOF. Suppose $\mathbf{F}^1 \neq \mathbf{F}$ is an extra-special subgroup of $\mathbf{W}$ of order 3^5. We claim $\mathrm{Z}(\mathbf{W}) \leq \mathbf{F}^1$. Otherwise $\mathrm{Z}(\mathbf{W}) \cap \mathbf{F}^1 = \{1\}$, and so $\mathbf{W} = \mathrm{Z}(\mathbf{W}) \times \mathbf{F}^1$, which is impossible. Thus $\mathrm{Z}(\mathbf{W})$ coincides with $\mathrm{Z}(\mathbf{F}^1)$, since the former is a central subgroup of $\mathbf{F}^1$ of order 3.

Now $\mathbf{F}^1\mathbf{F} = \mathbf{W}$, as $\mathbf{F}$ is a maximal subgroup of $\mathbf{W}$. Hence $\mathbf{F}^1 \cap \mathbf{F}$ has order $3^{5+5-6} = 3^4$. Moreover $\mathrm{Z}(\mathbf{W}) \leq \mathbf{F}^1 \cap \mathbf{F}$. So $\overline{\mathbf{F}}^1 = \mathbf{F}^1/\,\mathrm{Z}(\mathbf{W})$ is an elementary abelian subgroup of $\overline{\mathbf{W}} = \mathbf{W}/\,\mathrm{Z}(\mathbf{W})$ of order 3^4, distinct from $\overline{\mathbf{F}}$.

Let $\overline{\mathbf{S_L}}$ denote the subgroup $\mathbf{S_L}\,\mathrm{Z}(\mathbf{W})/\,\mathrm{Z}(\mathbf{W})$ of $\overline{\mathbf{W}}$. Then $\overline{\mathbf{W}} = \overline{\mathbf{F}} \rtimes \overline{\mathbf{S_L}}$. Pick $x \in \overline{\mathbf{F}}^1 \backslash \overline{\mathbf{F}}$. So we also have $\overline{\mathbf{W}} = \overline{\mathbf{F}} \rtimes \langle x \rangle$. Since $\overline{\mathbf{F}}$ is abelian, the actions of $\overline{\mathbf{S_L}}$ and $\langle x \rangle$ on $\overline{\mathbf{F}}$ coincide. But $\mathrm{C}_{\overline{\mathbf{F}}}(\overline{\mathbf{S_L}}) = \mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S_L}) = \mathbf{W}'/\,\mathrm{Z}(\mathbf{W})$ has order 3^2, while $\mathrm{C}_{\overline{\mathbf{F}}}(\langle x \rangle) \geq \overline{\mathbf{F}} \cap \overline{\mathbf{F}}^1$ has order $\geq 3^3$. This contradiction means that no such group $\mathbf{F}^1$ exists. $\square$

COROLLARY 3.1.10. *$\mathbf{E}$ is the unique subgroup of $\mathbf{W}$ isomorphic to $\mathbb{Z}_3^4$.*

PROOF. Let $\mathbf{E}^1$ be a subgroup of $\mathbf{W}$ isomorphic to $\mathbb{Z}_3^4$. Then $\mathbf{E}^1 \not\leq \mathbf{F}$. So $\mathbf{E}^1 \cap \mathbf{F}$ is an abelian subgroup of $\mathbf{F}$ of order $3^3 = 3^{5+4-6}$. Since $\mathbf{F} \cong 3^{1+4}$, this implies that $\mathrm{Z}(\mathbf{F}) \leq \mathbf{E}^1 \cap \mathbf{F}$.

Let $x \in \mathbf{E}^1 \backslash \mathbf{F}$. Then $\langle x \rangle \equiv \mathbf{S_L}$ modulo $\mathbf{F}$. Since $\overline{\mathbf{F}}$ is abelian, this implies that the actions of $\langle x \rangle$ and $\mathbf{S_L}$ on $\overline{\mathbf{F}}$ are identical. So

$$(\mathbf{E}^1 \cap \mathbf{F})/\,\mathrm{Z}(\mathbf{F}) \leq \mathrm{C}_{\overline{\mathbf{F}}}(x) = \mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S_L}) = \mathbf{W}'/\,\mathrm{Z}(\mathbf{F}).$$

Since the first and last terms have order 3^2, we conclude that $\mathbf{E}^1 \cap \mathbf{F} = \mathbf{W}'$. Then $\mathbf{E}^1 \leq \mathrm{C}_{\mathbf{W}}(\mathbf{W}') = \mathbf{E}$. So $\mathbf{E}^1 = \mathbf{E}$. $\square$

LEMMA 3.1.11. $\mathrm{N}_{\mathbf{M_c}}(\mathbf{E}) = \mathbf{E} \rtimes \mathbf{M}$, *where* $\mathbf{M} \cong \mathbf{M}_{10}$ *can be chosen to contain* $\mathbf{Q}$. *Let* $\mathbf{S_M} = \mathbf{M} \cap \mathbf{W}$ *and* $\mathbf{N_M} = \mathrm{N}_{\mathbf{M}}(\mathbf{S_M})$. *Then* $\mathbf{S_M}$ *is a Sylow* 3*-subgroup of* $\mathbf{M}$ *while* $\mathbf{N_M} = \mathbf{S_M} \rtimes \mathbf{Q}$ *and*

$$\mathrm{N}_{\mathbf{M_c}}(\mathbf{W}) = \mathbf{W} \rtimes \mathbf{Q} = \mathbf{F} \rtimes \mathbf{N_L} = \mathbf{E} \rtimes \mathbf{N_M} = (\mathbf{F} \rtimes \mathbf{L}) \cap (\mathbf{E} \rtimes \mathbf{M}).$$

PROOF. It follows from Lemma 3.1.6 that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{W}) \leq \mathrm{N}_{\mathbf{M_c}}\big(\mathrm{Z}(\mathbf{F})\big) = \mathbf{F} \rtimes \mathbf{L}$. So $\mathrm{N}_{\mathbf{M_c}}(\mathbf{W}) = \mathbf{F} \rtimes \mathrm{N}_{\mathbf{L}}(\mathbf{W}) = \mathbf{F} \rtimes (\mathbf{S_L} \rtimes \mathbf{Q}) = \mathbf{F} \rtimes \mathbf{N_L}$.

From the Atlas $\mathrm{N}_{\mathbf{M_c}}(\mathbf{E}) = \mathbf{E} \rtimes \mathbf{M}$, where $\mathbf{M} \cong \mathbf{M}_{10}$. Clearly $\mathbf{W} \leq \mathrm{N}_{\mathbf{M_c}}(\mathbf{E})$. Let $\mathbf{S_M} = \mathbf{M} \cap \mathbf{W}$. Then $\mathbf{S_M}$ is a Sylow 3-subgroup of $\mathbf{M}$. Let $\mathbf{N_M} = \mathrm{N}_{\mathbf{M}}(\mathbf{S_M})$. Then $\mathbf{N_M} = \mathbf{S_M} \rtimes \mathbf{Q}^1$, where $\mathbf{Q}^1 \cong Q_8$. Clearly $\mathbf{Q}^1 \leq \mathrm{N}_{\mathbf{M_c}}(\mathbf{W})$.

From the first paragraph it follows that $\mathbf{Q}$ and $\mathbf{Q}^1$ are Sylow 2-subgroups of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{W})$. Hence $(\mathbf{Q}^1)^x = \mathbf{Q}$, for some $x \in \mathbf{W}$. We replace $\mathbf{M}$ by $\mathbf{M}^x$ and $\mathbf{S_M}$ by $(\mathbf{S_M})^x$. Then we still have $\mathrm{N}_{\mathbf{M_c}}(\mathbf{E}) = \mathbf{E} \rtimes \mathbf{M}$. Also $\mathbf{N_M} = \mathbf{S_M} \rtimes \mathbf{Q}$ and $\mathrm{N}_{\mathbf{M_c}}(\mathbf{W}) = \mathbf{E} \rtimes \mathbf{N_M}$.

Finally $(\mathbf{E} \rtimes \mathbf{M}) \cap (\mathbf{F} \rtimes \mathbf{L}) \leq \mathrm{N}_{\mathbf{M_c}}(\mathbf{W})$, since $\mathbf{W} = \mathbf{EF}$. But the opposite inclusion follows from the first and third paragraphs of this proof. Hence $\mathrm{N}_{\mathbf{M_c}}(\mathbf{W}) = (\mathbf{F} \rtimes \mathbf{L}) \cap (\mathbf{E} \rtimes \mathbf{M})$. $\square$

Note that the Sylow 3-subgroup $\mathbf{S}_{\mathbf{M}}$ of $\mathbf{M} \cong \mathbf{M}_{10}$ is isomorphic to $\mathbb{Z}_3^2$.

PROPOSITION 3.1.12. *$\mathbf{E} \rtimes \mathbf{M}$ has two orbits on $\mathbf{E}^{\#}$. One consists of 20 elements from the class $(3A)$ of $\mathbf{M_c}$ containing α, while the other consists of 60 elements from the class $(3B)$ of $\mathbf{M_c}$. Let $\beta \in \mathbf{E}$ come from the class $(3B)$ of $\mathbf{M_c}$. Then*

$$\begin{aligned} &\mathrm{C}_{\mathbf{M}}(\alpha) = \mathbf{S}_{\mathbf{M}} \rtimes \mathrm{Z}_4^{(1)} < \mathbf{S}_{\mathbf{M}} \rtimes \mathbf{Q} = \mathrm{N}_{\mathbf{M}}\big(\langle\alpha\rangle\big), \text{ where } \mathrm{Z}_4^{(1)} \cong \mathbb{Z}_4, \\ &\mathrm{C}_{\mathbf{M}}(\beta) = \mathrm{A}_4 < \mathrm{S}_4 = \mathrm{N}_{\mathbf{M}}\big(\langle\beta\rangle\big) < \mathbf{M}' \cong \mathfrak{A}_6, \text{ where } \mathrm{A}_4 \cong \mathfrak{A}_4 \text{ and } \mathrm{S}_4 \cong \mathfrak{S}_4. \end{aligned} \tag{3.1.13}$$

Furthermore

$$\mathrm{C}_{\mathbf{M_c}}(\beta) = \mathbf{E} \rtimes \mathrm{A}_4 < \mathrm{N}_{\mathbf{M_c}}(\langle\beta\rangle) = \mathbf{E} \rtimes \mathrm{S}_4.$$

PROOF. Since $\alpha \in \mathbf{E}$, it follows from Lemmas 3.1.3 and 3.1.11 that $\mathrm{N}_{\mathbf{E}\rtimes\mathbf{M}}\big(\langle\alpha\rangle\big) = (\mathbf{F} \rtimes \mathbf{L}) \cap (\mathbf{E} \rtimes \mathbf{M}) = \mathbf{E} \rtimes (\mathbf{S}_{\mathbf{M}} \rtimes \mathbf{Q})$, and $\mathrm{C}_{\mathbf{E}\rtimes\mathbf{M}}(\alpha) = (\mathbf{F} \rtimes \mathbf{C}) \cap (\mathbf{E} \rtimes \mathbf{M}) = \mathbf{E} \rtimes (\mathbf{S}_{\mathbf{M}} \rtimes \mathrm{Z}_4^{(1)})$, where $\mathrm{Z}_4^{(1)} < \mathbf{Q} < \mathbf{M}$ is isomorphic to $\mathbb{Z}_4$. So α lies in an $\mathbf{M}$-orbit of length $|\mathbf{M} : \mathbf{S}_{\mathbf{M}} \rtimes \mathrm{Z}_4^{(1)}| = 20$.

Let β be an element of the $(3B)$ class of $\mathbf{M_c}$. Then $\mathrm{N}_{\mathbf{M_c}}\big(\langle\beta\rangle\big) \cong \mathbb{Z}_3^4 \rtimes \mathfrak{S}_4$, by [**GL83**, p55]. Hence by Corollary 3.1.10, we may assume that $\mathrm{N}_{\mathbf{M_c}}\big(\langle\beta\rangle\big) = \mathbf{E} \rtimes \mathrm{S}_4$, where $\mathrm{S}_4 \cong \mathfrak{S}_4$. But $\mathbf{E} = \mathrm{O}_3(\mathbf{E} \rtimes \mathrm{S}_4)$. So $\beta \in \mathbf{E}$. Moreover $\mathbf{E} \rtimes \mathrm{S}_4 \le \mathbf{E} \rtimes \mathbf{M} = \mathrm{N}_{\mathbf{M_c}}(\mathbf{E})$. So, replacing S_4 by $(\mathbf{E} \rtimes \mathrm{S}_4) \cap \mathbf{M}$ if necessary, we may also assume that $\mathrm{S}_4 \le \mathbf{M}$. From the maximal subgroups of $\mathbf{M} \cong \mathbf{M}_{10}$ listed on page 4 of the Atlas it follows that any subgroup of $\mathbf{M}$ isomorphic to $\mathfrak{S}_4$ is contained in $\mathbf{M}' \cong \mathfrak{A}_6$. Choose an involution $\iota \in \mathrm{C}_{\mathbf{M}}(\beta)$. Such an involution exists because $\mathrm{C}_{\mathbf{M}}(\beta)$ is a subgroup of index at most 2 in $\mathrm{N}_{\mathbf{M}}\big(\langle\beta\rangle\big) \cong \mathfrak{S}_4$. All 3-elements of $\mathrm{C}_{\mathbf{M_c}}(\iota) \cong 2.\mathfrak{A}_8$ are real. So β is real in $\mathbf{M_c}$.

Hence $C_{\mathbf{M_c}}(\beta) = \mathbf{E} \rtimes A_4$, where $A_4 = {S_4}' \cong \mathfrak{A}_4$ is the unique subgroup of index 2 in S_4. We conclude that β lies in an $\mathbf{M}$-orbit of length $|\mathbf{M} : A_4| = 60$.

The $\mathbf{M}$-orbits of α and β account for all elements of $\mathbf{E}^{\#}$. $\square$

For the rest of the thesis β will denote an element of $\mathbf{E}$ which is a member of the class $(3B)$ of $\mathbf{M_c}$.

PROPOSITION 3.1.14. $\mathbf{L}$ *has a single orbit on* $\mathbf{F} \setminus Z(\mathbf{F})$, *consisting of* 240 *elements from the class* $(3B)$ *of* $\mathbf{M_c}$. *Hence* $\mathbf{L}$ *acts transitively on* $\overline{\mathbf{F}}^{\#}$.

PROOF. By Proposition 3.1.12 the group $\mathbf{E}$ contains only 20 elements of the class $(3A)$ of $\mathbf{M_c}$. But $|\mathbf{F} \cap \mathbf{E}| = 3^3$. So $\mathbf{F} \cap \mathbf{E}$ intersects the class $(3B)$ of $\mathbf{M_c}$. We suppose that the element β in Proposition 3.1.12 lies in $(\mathbf{F} \cap \mathbf{E}) \setminus Z(\mathbf{F})$.

From the Atlas $|C_{\mathbf{M_c}}(\beta)| = 972 = 3^5 \cdot 2^2$, and by Proposition 3.1.4 a Sylow 2-subgroup of $C_{\mathbf{F} \rtimes \mathbf{L}}(\beta)$ is trivial. Thus $|C_{\mathbf{F} \rtimes \mathbf{L}}(\beta)| \leq 3^5$. So β has at least $|\mathbf{F} \rtimes \mathbf{L}|/3^5 = 240$ distinct $\mathbf{F} \rtimes \mathbf{L}$-conjugates. But no element of $Z(\mathbf{F})$ is conjugate to β. Hence the $\mathbf{F} \rtimes \mathbf{L}$-orbit of β is contained in $\mathbf{F} \setminus Z(\mathbf{F})$. Since this latter set is of cardinality 240, we conclude that $\mathbf{F} \setminus Z(\mathbf{F})$ is the $\mathbf{F} \rtimes \mathbf{L}$-orbit of β.

Now $\overline{\mathbf{F}}$ is abelian. So $\mathbf{L}$ acts transitively on $\overline{\mathbf{F}}^{\#}$. $\square$

COROLLARY 3.1.15. *Let* $A_6 = \mathbf{M}'$. *Then* $C_{\mathbf{E}}(A_6) = \{1\}$. *Hence* $\mathbf{M}$ *acts faithfully on* $\mathbf{E}$. *Also* $\mathbf{E}$ *is an irreducible* $\mathbf{M}$*-group.*

PROOF. The group $A_6 \cong \mathfrak{A}_6$ is the unique non-trivial proper normal subgroup of $\mathbf{M} \cong \mathbf{M}_{10}$. But it follows from (3.1.13) that $C_{\mathbf{E}}(A_6) = \{1\}$. We conclude that $\mathbf{M}$ acts faithfully on $\mathbf{E}$.

From Proposition 3.1.12 the group $\mathbf{M}$ has orbits of length $1, 20, 60$ on $\mathbf{E}$. Hence $\mathbf{M}$ acts irreducibly on $\mathbf{E}$. $\square$

3.2. The Radical 3-chains of $\mathbf{M_c}$

We find the radical 3-subgroups of $\mathbf{M_c}$ using the following lemma.

LEMMA 3.2.1. *If* X *is a* $\mathbb{Z}_3^n$ *subgroup of* $\mathbf{M_c}$, *for* $n \geq 1$, *then* $\mathrm{N}_{\mathbf{M_c}}(\mathrm{X})$ *is conjugate to a subgroup of* $\mathbf{F} \rtimes \mathbf{L}$ *or* $\mathbf{E} \rtimes \mathbf{M}$.

PROOF. The result follows from [**Fk73**, 5.6, p70]. $\square$

We can now give the radical 3-subgroups of $\mathbf{M_c}$.

PROPOSITION 3.2.2. *Let* X *be a radical* 3*-subgroup of* $\mathbf{M_c}$. *Then* X *is conjugate in* $\mathbf{M_c}$ *to exactly one of*

$$\{1\}, \quad \mathbf{E}, \quad \mathbf{F} \quad \textit{or} \quad \mathbf{W}.$$

Furthermore any radical 3*-subgroup of* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{E})$ *is* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{E})$*-conjugate to* $\mathbf{E}$ *or* $\mathbf{W}$, *and any radical* 3*-subgroup of* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{F})$ *is* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{F})$*-conjugate to* $\mathbf{F}$ *or* $\mathbf{W}$.

PROOF. Assume $\mathrm{X} \neq \{1\}$. We have $\mathrm{N}_{\mathbf{M_c}}(\mathrm{X}) \leq \mathrm{N}_{\mathbf{M_c}}\big(\Omega\mathrm{Z}(\mathrm{X})\big)$. So the normalizer $\mathrm{N}_{\mathbf{M_c}}(\mathrm{X})$ is either contained in a conjugate of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{E}) = \mathbf{E} \rtimes \mathbf{M}$ or of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{F}) = \mathbf{F} \rtimes \mathbf{L}$, by Lemma 3.2.1. Hence we may assume that X is a radical 3-subgroup of one of these groups.

Suppose X is a radical 3-subgroup of $\mathbf{F} \rtimes \mathbf{L}$. Then $\mathrm{X} \geq \mathbf{F}$, since $\mathbf{F} = \mathrm{O}_3(\mathbf{F} \rtimes \mathbf{L})$. Now X is contained in some Sylow 3-subgroup $\mathbf{W}^1$ of $\mathbf{F} \rtimes \mathbf{L}$. Since $\mathbf{F}$ is maximal in $\mathbf{W}^1$, either $\mathrm{X} = \mathbf{F}$ or $\mathrm{X} = \mathbf{W}^1$ is conjugate to $\mathbf{W}$ in $\mathbf{F} \rtimes \mathbf{L}$.

Suppose on the other hand that X is a radical 3-subgroup of $\mathbf{E} \rtimes \mathbf{M}$. Then $\mathrm{X} \geq \mathbf{E}$, since $\mathbf{E} = \mathrm{O}_3(\mathbf{E} \rtimes \mathbf{M})$. If $\mathrm{X} \neq \mathbf{E}$, then $\mathrm{X} = \mathbf{E} \rtimes \mathrm{X_R}$, where $\mathrm{X_R} = \mathrm{X} \cap \mathbf{M}$ is a non-trivial radical 3-subgroup of $\mathbf{M}$. It follows that $\mathrm{N}_{\mathbf{E}\rtimes\mathbf{M}}(\mathrm{X}) = \mathbf{E} \rtimes \mathrm{N}_{\mathbf{M}}(\mathrm{X}) = \mathbf{E} \rtimes \mathrm{N}_{\mathbf{M}}(\mathrm{X_R})$. From the Atlas, $\mathrm{C}_{\mathbf{M}}(\mathrm{X_R})$ is a Sylow 3-subgroup of $\mathbf{M}$. Hence $\mathrm{X_R} = \mathrm{O}_3\big(\mathrm{N}_{\mathbf{M}}(\mathrm{X_R})\big) = \mathrm{C}_{\mathbf{M}}(\mathrm{X_R})$ is an $\mathbf{M}$-conjugate of $\mathbf{S_M}$. Thus $\mathrm{X} = \mathbf{E} \rtimes \mathrm{X_R}$ is conjugate to $\mathbf{W}$ in $\mathbf{M}$. □

COROLLARY 3.2.3. *Every radical* 3*-chain of* $\mathbf{M_c}$ *is conjugate to exactly one of*

$$\{1\}, \quad \{1\} < \mathbf{E}, \quad \{1\} < \mathbf{E} < \mathbf{W}, \quad \{1\} < \mathbf{F}, \quad \{1\} < \mathbf{F} < \mathbf{W} \quad \textit{or} \quad \{1\} < \mathbf{W}.$$

Furthermore, each of these 3*-chains is a radical* 3*-chain of* $\mathbf{M_c}$*.*

PROOF. This follows immediately from the previous proposition. □

Three of the radical 3-chains of $\mathbf{M_c}$ have the same stabilizers, as we now show.

LEMMA 3.2.4. *The normalizer of each of the radical* 3*-chains*

$$\{1\} < \mathbf{W}, \quad \{1\} < \mathbf{F} < \mathbf{W}, \quad \textit{and} \quad \{1\} < \mathbf{E} < \mathbf{W}$$

in either $\mathbf{M_c}$ *or* $\mathbf{M_c}.2$ *is the same as the normalizer of* $\mathbf{W}$ *in that group.*

PROOF. The groups $\mathbf{E}$ and $\mathbf{F}$ are characteristic subgroups of $\mathbf{W}$, by Corollaries 3.1.9 and 3.1.10. Thus the normalizer of $\mathbf{W}$ in $\mathbf{M_c}$ or $\mathbf{M_c}.2$ also normalizes each of the three given radical 3-chains. But since $\mathbf{W}$ occurs in each of the chains, the normalizer of any of the chains is contained in the normalizer of $\mathbf{W}$. This proves the lemma. □

Using (3.1.1), Lemmas 3.1.11 and 3.2.4 above, and Lemmas 3.7.1 and 3.7.2 below, we obtain the following descriptions of the normalizers of the radical 3-chains in $\mathbf{M_c}$ and $\mathbf{M_c}.2$:

Chain C	Chain Description	$\mathrm{N}_{\mathbf{M_c}}(C)$	$\mathrm{N}_{\mathbf{M_c}.2}(C)$	Parity
C_1	$\{1\}$	$\mathbf{M_c}$	$\mathbf{M_c}.2$	$+$
C_2	$\{1\} < \mathbf{E}$	$\mathbf{E} \rtimes \mathbf{M}$	$\mathbf{E} \rtimes (\mathbf{M} \times \langle c \rangle)$	$-$
C_3	$\{1\} < \mathbf{E} < \mathbf{W}$	$\mathbf{W} \rtimes \mathbf{Q}$	$\mathbf{W} \rtimes (\langle d \rangle \mathbf{Q})$	$+$
C_4	$\{1\} < \mathbf{F}$	$\mathbf{F} \rtimes \mathbf{L}$	$\mathbf{F} \rtimes (\langle d \rangle \mathbf{L})$	$-$
C_5	$\{1\} < \mathbf{F} < \mathbf{W}$	$\mathbf{W} \rtimes \mathbf{Q}$	$\mathbf{W} \rtimes (\langle d \rangle \mathbf{Q})$	$+$
C_6	$\{1\} < \mathbf{W}$	$\mathbf{W} \rtimes \mathbf{Q}$	$\mathbf{W} \rtimes (\langle d \rangle \mathbf{Q})$	$-$

TABLE 3.1. The Radical 3-chains of $\mathbf{M_c}$

3.3. The Character Degrees of $\mathbf{F} \rtimes \mathbf{L}$

Since $\mathbf{L} \cong 2.\mathfrak{S}_5$, we have from the Atlas

$$\begin{aligned} &\operatorname{Deg}\big((\mathbf{F} \rtimes \mathbf{L})/\mathbf{F}\big) = \{1^2, 4^5, 5^2, 6^3\}, \\ &\operatorname{Def}_3(\mathbf{F} \rtimes \mathbf{L} \bmod \mathbf{F}) = \{6^9, 5^3\}. \end{aligned} \tag{3.3.1}$$

The group $\mathbf{L}$ acts transitively on $\overline{\mathbf{F}}^{\#}$ by Proposition 3.1.14. So by Lemma 1.2.19, it also acts transitively on $\operatorname{Irr}(\overline{\mathbf{F}})^{\#}$. Hence any non-trivial linear character ψ of $\overline{\mathbf{F}}$ has 80 distinct $\mathbf{L}$-conjugates and $\mathrm{I}_{\mathbf{L}}(\psi)$ is cyclic of order 3. It follows from Theorem 1.2.10 that ψ extends to the cyclic group $\mathrm{I}_{\mathbf{L}}(\psi)$. Let $\tilde{\psi}$ be one such extension. Then from

Theorem 1.2.16 we have $\mathrm{Irr}(\overline{\mathbf{F}} \rtimes \mathbf{L} \mid \psi) = \left\{(\omega\tilde{\psi})^{\overline{\mathbf{F}}\rtimes\mathbf{L}} \mid \omega \in \mathrm{Irr}(\mathbb{Z}_3)\right\}$. Each $(\omega\tilde{\psi})^{\overline{\mathbf{F}}\rtimes\mathbf{L}}$ is of degree $\deg(\omega)\cdot\deg(\tilde{\psi})\cdot|\mathbf{L} : \mathrm{I}_{\mathbf{L}}(\psi)| = 80$. Thus

$$\mathrm{Deg}\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathbf{F}/\,\mathrm{Z}(\mathbf{F})\big) = \{80^3\}, \qquad \mathrm{Def}_3\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathbf{F}/\,\mathrm{Z}(\mathbf{F})\big) = \{6^3\}. \tag{3.3.2}$$

The group $\mathbf{F}$ is extra-special of order 3^5. So by Theorem 1.2.18 it has two characters of degree 9 which are non-trivial on $\mathrm{Z}(\mathbf{F})$. Moreover these characters are determined by their restrictions to $\mathrm{Z}(\mathbf{F})$. By Lemma 3.1.3 this implies that they form a single $\mathbf{F} \rtimes \mathbf{L}$-orbit. Let χ be one of these characters. Then $\mathrm{I}_{\mathbf{L}}(\chi) = \mathrm{C}_{\mathbf{L}}\big(\mathrm{Z}(\mathbf{F})\big) = \mathbf{C}$. Since a Sylow 3-subgroup of $\mathbf{C}$ is cyclic, χ extends to $\mathrm{I}_{\mathbf{L}}(\chi) = \mathbf{F} \rtimes \mathbf{C}$. Let $\tilde{\chi}$ be one such extension. Then $\mathrm{Irr}(\mathbf{F} \rtimes \mathbf{L} \mid \chi) = \big\{(\omega\tilde{\chi})^{\mathbf{F}\rtimes\mathbf{L}} \mid \omega \in \mathrm{Irr}(\mathbf{C})\big\}$. Since $\mathrm{Deg}(\mathbf{C}) = \mathrm{Deg}(2.\mathfrak{A}_5) = \{1, 2^2, 3^2, 4^2, 5, 6\}$, and $\mathrm{Deg}\big((\omega\tilde{\chi})^{\mathbf{F}\rtimes\mathbf{L}}\big) = \deg(\omega)\cdot\deg(\tilde{\chi})\cdot 2 = 18\cdot\deg(\omega)$, we have

$$\begin{aligned} \mathrm{Deg}\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathrm{Z}(\mathbf{F})\big) &= \{18, 36^2, 54^2, 72^2, 90, 108\}, \\ \mathrm{Def}_3\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathrm{Z}(\mathbf{F})\big) &= \{4^6, 3^3\}. \end{aligned} \tag{3.3.3}$$

From equations (3.3.1), (3.3.2) and (3.3.3) we obtain

$$\begin{aligned} \mathrm{Deg}(\mathbf{F} \rtimes \mathbf{L}) &= \{1^2, 4^5, 5^2, 6^3, 18, 36^2, 54^2, 72^2, 80^3, 90, 108\}, \\ \mathrm{Def}_3(\mathbf{F} \rtimes \mathbf{L}) &= \{6^{12}, 5^3, 4^6, 3^3\}. \end{aligned} \tag{3.3.4}$$

This allows us to prove the following

PROPOSITION 3.3.5. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_4) = \mathbf{F} \rtimes \mathbf{L}$ *has a unique* 3*-block, which necessarily induces the principal* 3*-block,* B_0*, of* $\mathbf{M_c}$*. Thus*

$$
\begin{aligned}
&\mathrm{k}(C_4, \mathsf{B}_0, 6) = 12, && \mathrm{k}(C_4, \mathsf{B}_0, 5) = 3,\\
&\mathrm{k}(C_4, \mathsf{B}_0, 4) = 6, && \mathrm{k}(C_4, \mathsf{B}_0, 3) = 3, \qquad (3.3.6)\\
&\mathrm{k}(C_4, \mathsf{B}_0, d) = 0, && \textit{for all other values of } d.
\end{aligned}
$$

PROOF. Since $\mathrm{C}_{\mathbf{F}\rtimes\mathbf{L}}(\mathbf{F}) = \mathrm{Z}(\mathbf{F}) \le \mathbf{F}$, the group $\mathbf{F} \rtimes \mathbf{L}$ has only one 3-block, b_0, which must be its principal 3-block. So b_0 induces the principal 3-block, B_0, of $\mathbf{M_c}$. We now obtain (3.3.6) from (3.3.4). □

3.4. The Character Degrees of $\mathbf{W} \rtimes \mathbf{Q}$

Recall from Lemma 3.1.11 that $\mathbf{W}\rtimes\mathbf{Q} = \mathbf{F}\rtimes\mathbf{N_L}$. Since $\mathbf{Q}$ acts transitively on $\mathbf{S_L}^{\#}$, by Lemma 1.2.19 it also acts transitively on $\mathrm{Irr}(\mathbf{S_L})^{\#}$. Let ϕ be an element of $\mathrm{Irr}(\mathbf{S_L})^{\#}$. Then $\mathrm{I}_{\mathbf{N_L}}(\phi) \cong \mathbb{Z}_3\times\mathbb{Z}_4$. By Theorems 1.2.10 and 1.2.16 we have $\mathrm{Deg}(\mathbf{N_L} \mid \mathbf{S_L}) = \{2^4\}$. Also $\mathrm{Deg}(\mathbf{N_L}/\mathbf{S_L}) = \mathrm{Deg}(\mathbf{Q}) = \{1^4, 2\}$. Hence

$$
\mathrm{Deg}(\mathbf{N_L}) = \{1^4, 2^5\}, \qquad \mathrm{Def}_3(\mathbf{W} \rtimes \mathbf{Q} \bmod \mathbf{F}) = \{6^9\}. \qquad (3.4.1)
$$

Next we consider the group $\overline{\mathbf{F}} \rtimes \mathbf{N_L}$. We identify $\mathrm{Irr}(\overline{\mathbf{F}})$ with the dual group $\overline{\mathbf{F}}^*$ of $\overline{\mathbf{F}}$. The action of $\mathbf{N_L}$ on $\overline{\mathbf{F}}$ induces an action of $\mathbf{N_L}$ on $\overline{\mathbf{F}}^*$. Lemma 1.2.20 and Proposition 3.1.8 imply that $\mathrm{C}_{\overline{\mathbf{F}}^*}(\mathbf{S_L})$ has order 3^2. It then follows from Lemma 3.1.7 that $\mathbf{Q}$ acts regularly and transitively on $\mathrm{C}_{\overline{\mathbf{F}}^*}(\mathbf{S_L})^{\#}$. Hence $\mathrm{C}_{\overline{\mathbf{F}}^*}(\mathbf{S_L})^{\#}$ is a $\mathbf{N_L}$-orbit of length 8. If $\psi \in \overline{\mathbf{F}}^*$, then by Proposition 3.1.4, its centralizer $\mathrm{C}_{\mathbf{N_L}}(\psi)$ is a 3-group. But $\mathbf{S_L}$ is the unique non-trivial 3-subgroup of $\mathbf{N_L}$. Thus, if ψ is not centralized by

$\mathbf{S_L}$, it must have a trivial centralizer in $\mathbf{N_L}$, in which case it lies in an $\mathbf{N_L}$-orbit of length 24. We conclude that

$$\operatorname{Orb}\big(\mathbf{N_L}, (\overline{\mathbf{F}}^{*})^{\#}\big) = \{8, 24^3\}. \tag{3.4.2}$$

By Theorem 1.2.15 each irreducible character of $\overline{\mathbf{F}}$ extends to its stabilizer in $\mathbf{N_L}$. Using Theorem 1.2.16 we find

$$\begin{aligned} \operatorname{Deg}\big(\mathbf{W} \rtimes \mathbf{Q} \mid \mathbf{F}/\operatorname{Z}(\mathbf{F})\big) &= \{8^3, 24^3\}, \\ \operatorname{Def}_3\big(\mathbf{W} \rtimes \mathbf{Q} \mid \mathbf{F}/\operatorname{Z}(\mathbf{F})\big) &= \{6^3, 5^3\}. \end{aligned} \tag{3.4.3}$$

As noted in Section 3.3, the group $\mathbf{F}$ has two characters of degree 9 which are non-trivial on $\operatorname{Z}(\mathbf{F})$. Moreover these characters are determined by their restrictions to $\operatorname{Z}(\mathbf{F})$. By Lemma 3.1.3 this implies that they form a single $\mathbf{W} \rtimes \mathbf{Q}$-orbit. Let χ be one of these characters. Then $\operatorname{I}_{\mathbf{N_L}}(\chi) = \operatorname{C}_{\mathbf{N_L}}\big(\operatorname{Z}(\mathbf{F})\big) = \mathbf{S_L} \rtimes \langle b \rangle$. The character χ extends to $\tilde{\chi} \in \operatorname{Irr}\big(\mathbf{F} \rtimes \operatorname{I}_{\mathbf{N_L}}(\chi)\big)$, as $\mathbf{S_L} \rtimes \langle b \rangle$ has cyclic Sylow subgroups. So $\operatorname{Irr}\big(\mathbf{W} \rtimes \mathbf{Q} \mid \operatorname{Z}(\mathbf{F})\big) = \big\{ (\omega\tilde{\chi})^{\mathbf{W} \rtimes \mathbf{Q}} \mid \omega \in \operatorname{Irr}(\mathbf{S_L} \rtimes \langle b \rangle) \big\}$. Hence $\operatorname{Deg}\big(\mathbf{F} \rtimes \operatorname{I}_{\mathbf{N_L}}(\chi) \mid \chi\big) = 9 \cdot \operatorname{Deg}\big(\mathbf{S_L} \rtimes \langle b \rangle\big)$. Now b inverts $\mathbf{S_L}$ by Lemma 3.1.2. So $\operatorname{Deg}\big(\mathbf{F} \rtimes \operatorname{I}_{\mathbf{N_L}}(\chi) \mid \chi\big) = \{9^4, 18^2\}$. Then by Clifford theory

$$\begin{aligned} \operatorname{Deg}\big(\mathbf{W} \rtimes \mathbf{Q} \mid \operatorname{Z}(\mathbf{F})\big) &= \{18^4, 36^2\}, \\ \operatorname{Def}_3\big(\mathbf{W} \rtimes \mathbf{Q} \mid \operatorname{Z}(\mathbf{F})\big) &= \{4^6\}. \end{aligned} \tag{3.4.4}$$

From equations (3.4.1), (3.4.3) and (3.4.4) we have

$$\begin{aligned} \operatorname{Deg}(\mathbf{W} \rtimes \mathbf{Q}) &= \{1^4, 2^5, 8^3, 18^4, 24^3, 36^2\}, \\ \operatorname{Def}_3(\mathbf{W} \rtimes \mathbf{Q}) &= \{6^{12}, 5^3, 4^6\}. \end{aligned} \tag{3.4.5}$$

This allows us to prove the following

PROPOSITION 3.4.6. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_3) = \mathbf{W} \rtimes \mathbf{Q}$ *has a unique* 3*-block, which necessarily induces the principal* 3*-block,* B_0*, of* $\mathbf{M_c}$*. Hence*

$$\begin{aligned} &\mathrm{k}(C_3, \mathsf{B}_0, 6) = 12, && \mathrm{k}(C_3, \mathsf{B}_0, 5) = 3, \\ &\mathrm{k}(C_3, \mathsf{B}_0, 4) = 6, && \mathrm{k}(C_3, \mathsf{B}_0, 3) = 0, \\ &\mathrm{k}(C_3, \mathsf{B}_0, d) = 0, && \textit{for all other values of } d. \end{aligned} \tag{3.4.7}$$

PROOF. Since $\mathrm{C}_{\mathbf{W} \rtimes \mathbf{Q}}(\mathbf{W}) = \mathrm{Z}(\mathbf{W}) \leq \mathbf{W}$, the group $\mathbf{W} \rtimes \mathbf{Q}$ has only one 3-block. This block necessarily induces the principal block B_0 of $\mathbf{M_c}$. We now obtain (3.4.7) from (3.4.5). ☐

3.5. The Character Degrees of $\mathbf{E} \rtimes \mathbf{M}$

Let $\mathbf{E}^* = \mathrm{Hom}\big(\mathbf{E}, \mathrm{GF}(3)\big)$ be the dual group to $\mathbf{E}$. We identify $\mathbf{E}^*$ and $\mathrm{Irr}(\mathbf{E})$. Both $\mathbf{E}$ and $\mathbf{E}^*$ can be considered as $\mathrm{GF}(3)\mathbf{M}$-modules. By Proposition 3.1.12 the group $\mathbf{M}$ has two orbits on $\mathbf{E}^{\#}$, of lengths 20 and 60. So $\mathbf{M}$ has two orbits on $(\mathbf{E}^*)^{\#}$ also.

LEMMA 3.5.1. *There exists an increasing chain*

$$\mathrm{K} < \mathrm{A}_4 < \mathrm{S}_4 < \mathrm{A}_6 < \mathbf{M} \tag{3.5.2}$$

of subgroups of $\mathbf{M}$*, with* $\mathrm{A}_6 = \mathbf{M}' \cong \mathfrak{A}_6$*,* $\mathrm{S}_4 \cong \mathfrak{S}_4$*,* $\mathrm{A}_4 = \mathrm{S}_4{}' \cong \mathfrak{A}_4$ *and* $\mathrm{K} = \mathrm{O}_2(\mathrm{A}_4) \cong \mathbb{Z}_2^2$*. The* $\mathrm{GF}(3)$*-vector space* $\mathbf{E}^*$ *is isomorphic to the regular* $\mathrm{GF}(3)\mathrm{K}$*-module. Hence* $\mathbf{M}$ *acts absolutely irreducibly on* $\mathbf{E}$*.*

PROOF. From the Atlas the derived group $\mathrm{A}_6 \cong \mathfrak{A}_6$ of $\mathbf{M}$ contains two conjugacy classes of maximal subgroups isomorphic to $\mathfrak{S}_4$, and these two classes fuse in $\mathbf{M}$. We choose a member S_4 of one of these conjugacy classes. We let A_4 denote the derived group $\mathrm{S}_4{}' \cong \mathfrak{A}_4$ of S_4, and let K denote $\mathrm{O}_2(\mathrm{A}_4) \cong \mathbb{Z}_2^2$. The existence of (3.5.2) now follows.

We claim that $\mathbf{E}^*$ is the regular GF(3)K-module. Since K is an elementary abelian 2-group, $\mathbf{E}^* \cong \mathbf{E}_1 \oplus \mathbf{E}_2 \oplus \mathbf{E}_3 \oplus \mathbf{E}_4$ as GF(3)K-modules, where each $\mathbf{E}_i$ is a one-dimensional GF(3)K-submodule of $\mathbf{E}^*$. Let ι be an involution in K. Since A_6 is simple and acts faithfully on $\mathbf{E}$, we can regard K as a subgroup of $\mathrm{SL}(4,3)$. So ι inverts an even, non-zero number of the $\mathbf{E}_i$. But $\mathrm{K} \cap \mathrm{Z}\big(\mathrm{SL}(4,3)\big) = \{1\}$, as $\mathrm{Z}(\mathrm{A}_6) = \{1\}$. So ι inverts exactly two of the A_i. The claim now follows easily.

By Corollary 3.1.15 the group $\mathbf{M}$ acts irreducibly on $\mathbf{E}$. So $\mathrm{End}_{\mathrm{GF}(3)\mathbf{M}}(\mathbf{E})$ is a subfield of the ring $\mathrm{End}_{\mathrm{GF}(3)\mathrm{K}}(\mathbf{E})$. But from the regularity of the GF(3)K-module $\mathbf{E}$, the ring $\mathbf{E} \cong \mathrm{GF}(3)\mathrm{K}$ is isomorphic to a direct sum of four copies of GF(3). Any subfield of such a direct sum is isomorphic to GF(3). The result now follows. $\square$

We can now describe the orbits of $\mathbf{M}$ on $\mathbf{E}^*$.

Lemma 3.5.3. *There exist ψ_1 and ψ_2 in $\mathbf{E}^*$ such that*

$$\begin{aligned} &\mathrm{I}_{\mathbf{M}}(\psi_1) = \mathbf{S}_{\mathbf{M}} \rtimes \mathrm{Z}_4^{(2)} < \mathbf{S}_{\mathbf{M}} \rtimes \mathbf{Q} = \mathrm{N}_{\mathbf{M}}\big(\langle \psi_1 \rangle\big), \quad \textit{where } \mathbf{Q} > \mathrm{Z}_4^{(2)} \cong \mathbb{Z}_4, \\ &\mathrm{I}_{\mathbf{M}}(\psi_2) = \mathrm{A}_4 < \mathrm{S}_4 = \mathrm{N}_{\mathbf{M}}\big(\langle \psi_2 \rangle\big), \quad \textit{where } \mathrm{A}_4 \cong \mathfrak{A}_4 \textit{ and } \mathrm{S}_4 \cong \mathfrak{S}_4. \end{aligned} \tag{3.5.4}$$

Hence

$$\mathrm{Orb}(\mathbf{M}, \mathbf{E}^*) = \{1, 20, 60\}. \tag{3.5.5}$$

PROOF. It follows from Proposition 3.1.8 that the GF(3)$\mathbf{M}$-module $\mathbf{W}' = \mathbf{E} \cap \mathbf{F}$ has codimension 1 in $\mathbf{E}$. Since $\mathbf{W} = \mathbf{E} \rtimes \mathbf{S_M}$, and $\mathbf{E}$ and $\mathbf{S_M}$ are abelian, it follows that $\mathbf{W}' = [\mathbf{E}, \mathbf{S_M}]$. So $\mathrm{C}_{\mathbf{E}^*}(\mathbf{S_M})$ has dimension 1, by Lemma 1.2.20.

Let ψ_1 be a generator of $\mathrm{C}_{\mathbf{E}^*}(\mathbf{S_M})$. Since $\mathbf{Q}$ normalizes $\mathbf{S_M}$, it must stabilize $\langle \psi_1 \rangle$. But it follows easily from Proposition 3.1.12 that $\mathrm{C}_{\mathbf{E}}(\mathbf{Q}) = \{1\}$. So $\mathrm{C}_{\mathbf{E}^*}(\mathbf{Q}) = \{1\}$, using Lemma 1.2.20. Hence $\mathrm{I}_{\mathbf{N_M}}(\psi_1) = \mathbf{S_M} \rtimes \mathrm{Z}_4^{(2)}$, where $\mathrm{Z}_4^{(2)} \cong \mathbb{Z}_4$. But from the Atlas $\mathbf{N_M} = \mathbf{S_M} \rtimes \mathbf{Q}$ is a maximal subgroup of $\mathbf{M} \cong \mathbf{M}_{10}$. It follows from this and the irreducibility of $\mathbf{E}^*$ as a GF(3)$\mathbf{M}$-module (see Corollary 3.1.15) that $\mathbf{N_M}$ is the stabilizer of $\langle \psi_1 \rangle$ in $\mathbf{M}$. Hence $\mathrm{I}_{\mathbf{M}}(\psi_1) = \mathbf{S_M} \rtimes \mathrm{Z}_4^{(2)}$. We conclude that the $\mathbf{M}$-orbit of ψ_1 contains $720/36 = 20$ elements.

Lemma 3.5.1 shows that $\mathbf{E}^*$ is isomorphic to the regular GF(3)K-module. In particular $\dim_{\mathrm{GF}(3)}(\mathrm{C}_{\mathbf{E}^*}(\mathrm{K})) = 1$. Let ψ_2 be a generator of $\mathrm{C}_{\mathbf{E}^*}(\mathrm{K})$. Then $\langle \psi_2 \rangle$ is stabilized by S_4, since $\mathrm{K} \triangleleft \mathrm{S}_4$. Let D_8 be a Sylow 2-subgroup of S_4. Then we see from (3.1.13) that $\mathrm{C}_{\mathbf{E}}(\mathrm{D}_8) = \{1\}$. So $\mathrm{C}_{\mathbf{E}^*}(\mathrm{D}_8) = \{1\}$ by Lemma 1.2.20. Hence $\mathrm{C}_{\mathbf{E}^*}(\mathrm{S}_4) = \{1\}$. This implies that $\mathrm{I}_{\mathrm{S}_4}(\psi_2) = \mathrm{A}_4$. But S_4, A_6 and $\mathbf{M}$ are the only subgroups of $\mathbf{M}$ containing S_4. If A_6 stabilizes $\langle \psi_2 \rangle$, then it has an orbit of length 2 on $(\mathbf{E}^*)^\#$. This is impossible since A_6 is simple. So S_4 is precisely the stabilizer of $\langle \psi_2 \rangle$ in $\mathbf{M}$. Hence $\mathrm{A}_4 = \mathrm{I}_{\mathbf{M}}(\psi_2)$. We conclude that the $\mathbf{E} \rtimes \mathbf{M}$-orbit of ψ_2 contains 60 elements.

The 20 $\mathbf{M}$-conjugates of ψ_1 and the 60 $\mathbf{M}$-conjugates of ψ_2 account for all non-trivial elements of $\mathbf{E}^*$. $\square$

We consider the two non-trivial orbits of $\mathbf{M}$ on $\mathbf{E}^*$ in turn.

Let μ be a non-trivial linear character of $\mathbf{S_M}$. All involutions in $\mathrm{I}_{\mathbf{M}}(\psi_1)$ invert $\mathbf{S_M}$. So $\mathrm{I}_{\mathbf{S_M} \rtimes \mathrm{Z}_4^{(2)}}(\mu) = \mathbf{S_M}$. Hence $\mathrm{Deg}(\mathrm{I}_{\mathbf{M}}(\psi_1) \mid \mu) = \{4\}$. Since $\mathrm{I}_{\mathbf{M}}(\psi_1)$ has two orbits on $\mathbf{S_M}^{\#}$ we get $\mathrm{Deg}(\mathrm{I}_{\mathbf{M}}(\psi_1) \mid \mathbf{S_M}) = \{4^2\}$. Also $\mathrm{Deg}(\mathrm{I}_{\mathbf{M}}(\psi_1)/\mathbf{S_M}) = \{1^4\}$. Thus $\mathrm{Deg}\big(\mathrm{I}_{\mathbf{M}}(\psi_1)\big) = \{1^4, 4^2\}$.

Now ψ_1 extends to $\mathbf{E} \rtimes \mathrm{I}_{\mathbf{M}}(\psi_1)$ by Theorem 1.2.15. So $\mathrm{Deg}(\mathbf{E} \rtimes \mathrm{I}_{\mathbf{M}}(\psi_1) \mid \psi_1) = \{1^4, 4^2\}$. The $\mathbf{M}$-orbit of ψ_1 has length 20. So from Clifford theory we obtain

$$\text{(3.5.6)} \qquad \mathrm{Deg}(\mathbf{E} \rtimes \mathbf{M} \mid \psi_1) = \{20^4, 80^2\}, \qquad \mathrm{Def}_3(\mathbf{E} \rtimes \mathbf{M} \mid \psi_1) = \{6^6\}.$$

Since $\mathrm{I}_{\mathbf{M}}(\psi_2) = \mathrm{A}_4$, we obtain $\mathrm{Deg}(\mathbf{E} \rtimes \mathrm{I}_{\mathbf{M}}(\psi_2) \mid \psi_2) = \mathrm{Deg}(\mathrm{A}_4) = \{1^3, 3\}$. The $\mathbf{M}$-orbit of ψ_2 has length 60. It follows from Clifford theory that

$$\text{(3.5.7)} \qquad \mathrm{Deg}(\mathbf{E} \rtimes \mathbf{M} \mid \psi_2) = \{60^3, 180\}, \qquad \mathrm{Def}_3(\mathbf{E} \rtimes \mathbf{M} \mid \psi_2) = \{5^3, 4\}.$$

Since $\mathbf{M} \cong \mathbf{M}_{10}$, we obtain from the Atlas

$$\text{(3.5.8)} \quad \mathrm{Deg}\big((\mathbf{E} \rtimes \mathbf{M})/\mathbf{E}\big) = \{1^2, 9^2, 10^3, 16\}, \qquad \mathrm{Def}_3(\mathbf{E} \rtimes \mathbf{M} \bmod \mathbf{E}) = \{6^6, 4^2\}.$$

From (3.5.6), (3.5.7) and (3.5.8) we have

$$\text{(3.5.9)} \qquad \begin{aligned} \mathrm{Deg}(\mathbf{E} \rtimes \mathbf{M}) &= \{1^2, 9^2, 10^3, 16, 20^4, 60^3, 80^2, 180\}, \\ \mathrm{Def}_3(\mathbf{E} \rtimes \mathbf{M}) &= \{6^{12}, 5^3, 4^3\}. \end{aligned}$$

We now have enough information to prove

PROPOSITION 3.5.10. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_2) = \mathbf{E} \rtimes \mathbf{M}$ *has a unique* 3*-block, namely the principal* 3*-block. This block necessarily induces the principal* 3*-block,* B_0*, of* $\mathbf{M_c}$.

Thus

$$\text{(3.5.11)}\qquad \begin{array}{ll} \mathrm{k}(C_2, \mathsf{B}_0, 6) = 12, & \mathrm{k}(C_2, \mathsf{B}_0, 5) = 3, \\ \mathrm{k}(C_2, \mathsf{B}_0, 4) = 3, & \mathrm{k}(C_2, \mathsf{B}_0, 3) = 0, \\ \mathrm{k}(C_2, \mathsf{B}_0, d) = 0, & \textit{for all other values of } d. \end{array}$$

PROOF. The first statement follows from the fact that $\mathrm{C}_{\mathbf{E}\rtimes\mathbf{M}}(\mathbf{E}) = \mathbf{E}$. Then (3.5.11) comes from (3.5.9). ☐

3.6. The Ordinary Conjecture for the prime $p = 3$

From [**Con85**, p101], the group $\mathrm{N}_{\mathbf{M_c}}(C_1) = \mathbf{M_c}$ has three 3-blocks of defect 0, and the principal block B_0 of defect 6, which contains the remaining 21 characters. We list the characters of the principal block and their defects:

Character	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
Degree	1	22	231	252	770	770	896	896	1750	3520	3520
3-Defect	6	6	5	4	6	6	6	6	6	6	6

Character	χ_{12}	χ_{13}	χ_{15}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	
Degree	4500	4752	5544	8250	8250	9625	9856	9856	10395	10395	
3-Defect	4	3	4	5	5	6	6	6	3	3	

Thus

$$\mathrm{k}(C_1, \mathsf{B}_0, 6) = 12, \qquad \mathrm{k}(C_1, \mathsf{B}_0, 5) = 3,$$

$$\mathrm{k}(C_1, \mathsf{B}_0, 4) = 3, \qquad \mathrm{k}(C_1, \mathsf{B}_0, 3) = 3, \tag{3.6.1}$$

$$\mathrm{k}(C_1, \mathsf{B}_0, d) = 0, \qquad \text{for all other values of } d.$$

We now have enough information for the following theorem.

THEOREM 3.6.2. *The Ordinary Conjecture holds for McLaughlin's simple group and the prime* $p = 3$.

PROOF. From Conjecture 1.4.2 and Table 3.1 on page 48, the Ordinary Conjecture for the prime $p = 3$ asserts that

$$\mathrm{k}(C_1, \mathsf{B}_0, d) + \mathrm{k}(C_3, \mathsf{B}_0, d) = \mathrm{k}(C_2, \mathsf{B}_0, d) + \mathrm{k}(C_4, \mathsf{B}_0, d) \tag{3.6.3}$$

for all values of $d \in \mathbb{Z}$.

From (3.3.6), (3.4.7), (3.5.11) and (3.6.1) we obtain the following sums for the equation above for various values of d:

3-Defect	C_1		C_3		C_2		C_4
6	12	+	12	=	12	+	12
5	3	+	3	=	3	+	3
4	3	+	6	=	3	+	6
3	3	+	0	=	0	+	3

TABLE 3.2. The Ordinary Conjecture for $p = 3$

The summands in (3.6.3) are zero for all other values of d. This completes the proof of the theorem. $\square$

3.7. The Groups $\mathbf{E} \rtimes (\mathbf{M} \times \langle c \rangle)$ and $\mathbf{F} \rtimes (\langle d \rangle \mathbf{L})$

We begin with descriptions of the normalizers of $\mathbf{E}$ and $\mathbf{F}$ in $\mathbf{M_c}.2$.

LEMMA 3.7.1. $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{E}) = \mathbf{E} \rtimes (\mathbf{M} \times \langle c \rangle)$, *where c is an involution inverting* $\mathbf{E}$. *Hence* $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{W}) = \mathbf{W} \rtimes (\mathbf{Q} \times \langle c \rangle)$. *The group* $\mathbf{M} \times \langle c \rangle$ *acts faithfully on* $\mathbf{E}$.

PROOF. From the Atlas, $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{E})$ has the form $\mathbf{E} \rtimes (\mathbf{M} \times \langle c \rangle)$ for some involution c in $\mathbf{M_c}.2 \backslash \mathbf{M_c}$. centralizing $\mathbf{M} \cong \mathbf{M}_{10}$. From the order of $\mathrm{C}_{\mathbf{M_c}}(c)$ in the Atlas, it is clear that c acts as a non-trivial automorphism of the $\mathrm{GF}(3)\mathbf{M}$-module $\mathbf{E}$. But $\mathrm{End}_{\mathrm{GF}(3)\mathbf{M}}(\mathbf{E}) \cong \mathrm{GF}(3)$ by Corollary 3.5.1. So c inverts $\mathbf{E}$.

It is clear that c normalizes $\mathbf{W} = \mathbf{E} \rtimes \mathbf{S_M}$. So $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{W}) = \mathbf{W} \rtimes (\mathbf{Q} \times \langle c \rangle)$.

Since $\mathrm{Z}(\mathbf{M}) = \{1\}$, the embedding of $\mathbf{M}$ in $\mathrm{GL}(4,3) = \mathrm{GL}(\mathbf{E})$ does not contain $-\mathrm{I}$. If $x \in (\mathbf{M} \times \langle c \rangle) \backslash \mathbf{M}$ centralizes $\mathbf{E}$, then $xc \in \mathbf{M}$ inverts $\mathbf{E}$, contradicting the previous sentence. Hence $\mathbf{M} \times \langle c \rangle$ acts faithfully on $\mathbf{E}$. $\square$

LEMMA 3.7.2. $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{F}) = \mathbf{F} \rtimes (\langle d \rangle \mathbf{L})$, *where d is an element of* $\mathbf{M_c}.2$ *of order* 4. *The element d satisfies the equations* $d^2 = \tau$, $[d, \mathbf{C}] = \{1\}$ *and* $[d, a] = \tau$. *Hence* $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{W}) = \mathbf{W} \rtimes (\langle d \rangle \mathbf{Q})$. *The group* $\langle d \rangle \mathbf{L}$ *acts faithfully on* $\overline{\mathbf{F}}$.

PROOF. From [**Con85**] we have $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{F}) = \mathbf{F} \rtimes (\mathbf{L}.2)$, where $\mathbf{L}.2 \cong 4.\mathfrak{S}_5$. We denote by d a generator of the normal subgroup $\mathbb{Z}_4$ of $\mathbf{L}.2$. Then d^2 is the only generator τ of the unique normal subgroup $\langle \tau \rangle$ of order 2 in $\mathbf{L} \cong 2.\mathfrak{S}_5$.

Since $\mathbf{L}$ normalizes the cyclic group $\langle d \rangle$, its derived group $\mathbf{C}$ centralizes d. In particular d normalizes $\mathbf{W} = \mathbf{F} \rtimes \mathbf{S_L}$, since $\mathbf{S_L}$ is contained in $\mathbf{C}$. So $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{W}) = \mathbf{W} \rtimes (\langle d \rangle \mathbf{Q})$.

The Sylow 2-subgroups $\mathbf{Q} \times \langle c \rangle$ and $\langle d \rangle \mathbf{Q}$ of $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{W})$ are isomorphic. But $\mathrm{C}_{\mathbf{Q}\times\langle c \rangle}(\mathbf{Q}) = \langle \tau \rangle \times \langle c \rangle$ has exponent 2. So the 4-element $\langle d \rangle$ is not centralized by $\mathbf{Q}$. Hence a does not centralize d. We conclude that $d^a = d^{-1}$.

Let x be an element of $\mathrm{C}_{\langle d \rangle \mathbf{L}}(\overline{\mathbf{F}})$. Then $x^2 \in \mathbf{L}$ centralizes $\overline{\mathbf{F}}$. So $x^2 = 1$ and x lies in the class $(2B)$ of $\mathbf{M_c}.2$. But then x centralizes $\mathbf{F}$, since x is a $3'$-element and $\overline{\mathbf{F}} = \mathbf{F}/\Phi(\mathbf{F})$. This contradicts the fact that $|\mathbf{F}| = 3^5 \not| |\mathrm{C}_{\mathbf{M_c}}(x)|$. $\square$

Corollary 3.7.3. $\mathbf{Q} \times \langle c \rangle = \langle d \rangle \mathbf{Q}$. *So* $\langle d \rangle = \langle cb \rangle$, *and* d *inverts* $\mathrm{Z}(\mathbf{F})$.

Proof. The Sylow 2-subgroups $\mathbf{Q} \times \langle c \rangle$ and $\langle d \rangle \mathbf{Q}$ of $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{W})$ are conjugate by some $w \in \mathbf{W}$. Then $\mathbf{Q} = \langle d \rangle \mathbf{Q} \cap \mathbf{M_c} = \mathbf{Q}^w$. But $\mathbf{Q}$ does not centralize any element of $\mathbf{W}$, by Proposition 3.1.4. So $w = 1$. Hence $\mathbf{Q} \times \langle c \rangle = \langle d \rangle \mathbf{Q}$.

Now d centralizes $b \in \mathbf{C}$. So $d \in \mathrm{C}_{\mathbf{Q}\times\langle c \rangle}(b) = \langle b \rangle \times \langle c \rangle$. The latter group has exactly two cyclic subgroups $\langle b \rangle$ and $\langle cb \rangle$ of order 4. Since $\langle d \rangle$ is such a subgroup, and is not equal to $\langle b \rangle$, it must equal $\langle bc \rangle$. The involution c inverts $\mathrm{Z}(\mathbf{F})$, since by Lemma 3.7.1 it inverts $\mathbf{E}$ and $\mathrm{Z}(\mathbf{F}) = \mathrm{Z}(\mathbf{W})$ is a subgroup of $\mathbf{E}$. Also b centralizes $\mathrm{Z}(\mathbf{F})$, by Lemma 3.1.3. We conclude that d inverts $\mathrm{Z}(\mathbf{F})$. $\square$

3.8. The Invariant Characters of $\mathbf{E} \rtimes \mathbf{M}$

Let ψ_1 and ψ_2 be representatives of the two non-trivial orbits of $\mathbf{M}$ on $\mathbf{E}^*$, as in Lemma 3.5.3. Then ψ_1 and ψ_2 are not conjugate in $\mathbf{M} \times \langle c \rangle$ since their stabilizers in

$\mathbf{M}$ are not isomorphic. Hence they are representatives of the two non-trivial orbits of $\mathbf{M} \times \langle c \rangle$ on $\mathbf{E}^*$. Since c inverts $\mathbf{E}$, it also inverts $\mathbf{E}^*$.

From (3.5.4) there exists $x \in \mathbf{Q}$ which inverts ψ_1. Hence xc centralizes ψ_1. So $\mathrm{I}_{\mathbf{M}\times\langle c\rangle}(\psi_1) = \langle \mathrm{I}_{\mathbf{M}}(\psi_1), xc \rangle$. But c centralizes $\mathbf{M}$. So $\mathrm{I}_{\mathbf{M}\times\langle c\rangle}(\psi_1) \cong \mathrm{I}_{\mathbf{M}}(\psi_1)\langle x \rangle = \mathbf{S_M} \rtimes \mathbf{Q}$. This group is Frobenius. It follows easily that $\mathrm{Deg}\big(\mathrm{I}_{\mathbf{M}\times\langle c\rangle}(\psi_1)\big) = \{1^4, 2, 8\}$.

As $\mathbf{E}$ is abelian, Theorem 1.2.15 shows that ψ_1 extends to $\mathbf{E} \rtimes \mathrm{I}_{\mathbf{M}\rtimes\langle c\rangle}(\psi_1)$. Then by Clifford theory

$$(3.8.1) \qquad \mathrm{Deg}\big(\mathbf{E} \rtimes (\mathbf{M} \times \langle c \rangle) \mid \psi_1\big) = \{20^4, 40, 160\}.$$

We deduce from (3.5.6) and (3.8.1) that

$$(3.8.2) \qquad \mathrm{Inv}(\mathbf{E} \rtimes \mathbf{M} \mid \psi_1) = \{20^2\}, \qquad \mathrm{InvDef}_3(\mathbf{E} \rtimes \mathbf{M} \mid \psi_1) = \{6^2\}.$$

From (3.5.4) there exists $y \in \mathrm{S}_4\backslash\mathrm{A}_4$ which inverts ψ_2. Then, by an argument analogous to that used for ψ_1, we find that $\mathrm{I}_{\mathbf{M}\times\langle c\rangle}(\psi_2) \cong S_4$. Hence $\mathrm{Deg}\big(\mathrm{I}_{\mathbf{M}\times\langle c\rangle}(\psi_2)\big) = \{1^2, 2, 3^2\}$.

As ψ_2 also extends to its stabilizer in $\mathbf{E} \rtimes (\mathbf{M} \times \langle c \rangle)$, we have

$$(3.8.3) \qquad \mathrm{Deg}\big(\mathbf{E} \rtimes (\mathbf{M} \times \langle c \rangle) \mid \psi_2\big) = \{60^2, 120, 180^2\}.$$

Using (3.5.7) and (3.8.3) we obtain

$$(3.8.4) \qquad \mathrm{Inv}(\mathbf{E} \rtimes \mathbf{M} \mid \psi_2) = \{60, 180\}, \qquad \mathrm{InvDef}_3(\mathbf{E} \rtimes \mathbf{M} \mid \psi_2) = \{5, 4\}.$$

All characters of $\mathbf{M}$ are invariant in $\mathbf{M} \times \langle c \rangle$. This and (3.5.8) imply that

$$(3.8.5) \qquad \mathrm{Inv}(\mathbf{M}) = \{1^2, 9^2, 10^3, 16\}, \qquad \mathrm{InvDef}_3(\mathbf{E} \rtimes \mathbf{M} \bmod \mathbf{E}) = \{6^6, 4^2\}.$$

From (3.8.1), (3.8.3) and (3.8.5) we have

(3.8.6) $\mathrm{Inv}(\mathbf{E} \rtimes \mathbf{M}) = \{1^2, 9^2, 10^3, 16, 20^2, 60, 180\}, \quad \mathrm{InvDef}_3(\mathbf{E} \rtimes \mathbf{M}) = \{6^8, 5, 4^3\}.$

We can now prove the following

Proposition 3.8.7. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_2) = \mathbf{E} \rtimes \mathbf{M}$ *has a unique* 3*-block. This block induces of the principal* 3*-block,* B_0*, of* $\mathbf{M_c}$*. Moreover*

(3.8.8)
$$\begin{aligned}
&\mathrm{k}(\,C_2, \mathsf{B}_0, 6, \overline{\mathbf{M_c}.2}\,) = 8, &&\mathrm{k}(\,C_2, \mathsf{B}_0, 5, \overline{\mathbf{M_c}.2}\,) = 1,\\
&\mathrm{k}(\,C_2, \mathsf{B}_0, 4, \overline{\mathbf{M_c}.2}\,) = 3, &&\mathrm{k}(\,C_2, \mathsf{B}_0, 3, \overline{\mathbf{M_c}.2}\,) = 0,\\
&\mathrm{k}(\,C_2, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = 0, &&\textit{for all other values of } d.
\end{aligned}$$

Proof. This follows immediately from (3.8.6) and Proposition 3.5.10. ☐

3.9. The Invariant Characters of $\mathbf{F} \rtimes \mathbf{L}$

Lemma 3.9.1. *We have*

(3.9.2) $\mathrm{Inv}(\mathbf{L}) = \{1^2, 4^3, 5^2, 6\}, \qquad \mathrm{InvDef}_3(\mathbf{F} \rtimes \mathbf{L} \bmod \mathbf{F}) = \{6^7, 5\}.$

Proof. Let $\mu \in \mathrm{Irr}(\mathbf{L})$. Recall that $[d, \mathbf{C}] = 1$ and $[d, a] = \tau$. Hence $\mu^d(x) = \mu(x)$ for $x \in \mathbf{C}$, and $\mu^d(x) = \mu(\tau x)$ for $x \in \mathbf{L}\backslash\mathbf{C}$. Since τ is central in $\mathbf{L}$, there exists $\nu \in \mathrm{Irr}(\langle\tau\rangle)$ such that $\mu(\tau x) = \nu(\tau)\mu(x)$ for all $x \in \mathbf{L}\backslash\mathbf{C}$. But $\nu(\tau) = \pm 1$, since $\langle\tau\rangle \cong \mathbb{Z}_2$. So $\mu^d(x) = \pm\mu(x)$, depending on whether $\mu(\tau) = \pm\mu(1)$.

From the Atlas, we see that d stabilizes all characters of $\mathbf{L} \cong 2.\mathfrak{A}_5.2$, except the four extensions of the characters χ_8, χ_9 in the Atlas character table for $2.\mathfrak{A}_5$. The lemma now follows easily from that table. ☐

Let χ be one of the two characters of the extra-special 3-group $\mathbf{F}$ of degree 9. Since these two characters form a single $\mathbf{L}$-orbit, they also form a single $\mathbf{L}.2$-orbit. Hence $\mathrm{I}_{\mathbf{L}.2}(\chi)$ has index 2 in $\mathbf{L}.2$. Both a and d invert $\mathrm{Z}(\mathbf{F})$. So $\mathrm{I}_{\mathbf{L}.2}(\chi) = \mathbf{C}.\langle ad\rangle$. But d centralizes $\mathbf{C}$. Also $(ad)^2 = \tau = a^2$, since $a^2 = d^2 = [a,d] = \tau$. Hence $\mathrm{I}_{\mathbf{L}.2}(\chi) = \mathbf{C}.\langle ad\rangle \cong \mathbf{L} \cong 2.\mathfrak{S}_5$. A Sylow 3-Subgroup of $2.\mathfrak{S}_5$ is cyclic. So by Theorem 1.2.10, the character χ extends to $\mathbf{F} \rtimes \mathrm{I}_{\mathbf{L}.2}(\chi)$. Since $\mathrm{Deg}(2.\mathfrak{S}_5) = \{1^2, 4^5, 5^2, 6^3\}$, it follows from Theorem 1.2.16 that

$$\mathrm{Deg}\big(\mathbf{F} \rtimes (\mathbf{L}.2) \mid \mathrm{Z}(\mathbf{F})\big) = \{18^2, 72^5, 90^2, 108^3\} \tag{3.9.3}$$

Comparing (3.3.3) and (3.9.3) we have

$$\begin{aligned} &\mathrm{Inv}\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathrm{Z}(\mathbf{F})\big) = \{18, 72^2, 90, 108\}, \\ &\mathrm{InvDef}_3\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathrm{Z}(\mathbf{F})\big) = \{4^4, 3\}. \end{aligned} \tag{3.9.4}$$

Let $\psi \in (\overline{\mathbf{F}}^*)^\#$. By Proposition 3.1.14, the group $\mathbf{L}.2$ acts transitively on $\overline{\mathbf{F}}^\#$, and hence also on $(\overline{\mathbf{F}}^*)^\#$. So $\mathrm{I}_{\mathbf{L}.2}(\psi)$ has order 6. We may assume without loss of generality that $\mathbf{S}_\mathbf{L} \leq \mathrm{I}_{\mathbf{L}.2}(\psi)$.

It follows from Corollary 3.7.3 that $c \in \langle d\rangle\mathbf{Q}$ normalizes $\mathbf{F}$ (this could also be deduced from Corollary 3.1.9). So c acts on $\overline{\mathbf{F}} = \mathbf{F}/\mathrm{Z}(\mathbf{F})$ as well as on $\mathbf{S}_\mathbf{L} \leq \mathbf{E}$. Hence c acts on $\mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S}_\mathbf{L})$. By Proposition 3.1.8 the group $\mathbf{W}' = \mathbf{E} \cap \mathbf{F}$ is the inverse image of $\mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S}_\mathbf{L})$ in $\mathbf{F}$. So c inverts $\mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S}_\mathbf{L})$. Commutation in $\mathbf{F}$ induces a non-singular c-invariant bilinear form

$$\frac{\overline{\mathbf{F}}}{[\overline{\mathbf{F}}, \mathbf{S}_\mathbf{L}]} \times \mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S}_\mathbf{L}) \to \mathrm{Z}(\mathbf{F}).$$

As c inverts $\mathrm{C}_{\overline{\mathbf{F}}}(\mathbf{S_L})$ and $\mathrm{Z}(\mathbf{F})$, it necessarily centralizes $\overline{\mathbf{F}}/[\overline{\mathbf{F}}, \mathbf{S_L}]$. But $\mathrm{C}_{\overline{\mathbf{F}}^*}(\mathbf{S_L}) \cong \big(\overline{\mathbf{F}}/[\overline{\mathbf{F}}, \mathbf{S_L}]\big)^*$, by Lemma 1.2.20. So c centralizes $\mathrm{C}_{\overline{\mathbf{F}}^*}(\mathbf{S_L})$. We conclude that $\mathrm{I}_{\mathbf{L}.2}(\psi) = \mathbf{S_L} \rtimes \langle c \rangle \cong \mathfrak{S}_3$.

As $\overline{\mathbf{F}}$ is abelian, ψ extends to $\mathrm{I}_{\mathbf{F}\rtimes\mathbf{L}.2}(\psi)$. Then by Clifford theory

$$\operatorname{Deg}\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathbf{F}/\,\mathrm{Z}(\mathbf{F})\big) = \{80^2, 160\} \tag{3.9.5}$$

Comparing (3.3.2) and (3.9.5) we see that

$$\operatorname{Inv}\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathbf{F}/\,\mathrm{Z}(\mathbf{F})\big) = \{80\}, \qquad \operatorname{InvDef}_3\big(\mathbf{F} \rtimes \mathbf{L} \mid \mathbf{F} \bmod \mathrm{Z}(\mathbf{F})\big) = \{6\}. \tag{3.9.6}$$

From (3.9.2), (3.9.4), and (3.9.6), we have

$$\begin{aligned} \operatorname{Inv}(\mathbf{F} \rtimes \mathbf{L}) &= \{1^2, 4^3, 5^2, 6, 18, 72^2, 80, 90, 108\}, \\ \operatorname{InvDef}_3(\mathbf{F} \rtimes \mathbf{L}) &= \{6^8, 5, 4^4, 3\}. \end{aligned} \tag{3.9.7}$$

We can now prove

Proposition 3.9.8. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_4) = \mathbf{F} \rtimes \mathbf{L}$ *has a unique* 3*-block. This block induces the principal* 3*-block,* B_0*, of* $\mathbf{M_c}$*. Moreover*

$$\begin{aligned} &\mathrm{k}(\,C_4, \mathsf{B}_0, 6, \overline{\mathbf{M_c}.2}\,) = 8, && \mathrm{k}(\,C_4, \mathsf{B}_0, 5, \overline{\mathbf{M_c}.2}\,) = 1, \\ &\mathrm{k}(\,C_4, \mathsf{B}_0, 4, \overline{\mathbf{M_c}.2}\,) = 4, && \mathrm{k}(\,C_4, \mathsf{B}_0, 3, \overline{\mathbf{M_c}.2}\,) = 1, \\ &\mathrm{k}(\,C_4, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = 0, && \textit{for all other values of } d. \end{aligned} \tag{3.9.9}$$

Proof. This follows immediately from Proposition 3.3.6 and (3.9.7). ☐

3.10. The Invariant Characters of $\mathbf{W} \rtimes \mathbf{Q}$

We have $\mathrm{N}_{\mathbf{M}_c.2}(\mathbf{W}) = \mathbf{W} \rtimes \big(\mathbf{Q} \times \langle c\rangle\big) = \mathbf{F} \rtimes \big(\mathbf{N_L} \rtimes \langle c\rangle\big)$, where $\mathbf{N_L} = \mathrm{N}_{\mathbf{L}}(\mathbf{S_L}) = \mathbf{S_L} \rtimes \mathbf{Q}$. Choose a non-trivial character $\mu \in \mathrm{Irr}(\mathbf{S_L})$. Now $\mathrm{C}_{\mathbf{Q}}(\mathbf{S_L}) = \langle ab\rangle$, and both b and c invert $\mathbf{S_L}$. So $\mathrm{I}_{\mathbf{Q}\times\langle c\rangle}(\mu) = \langle ab, bc\rangle \cong Q_8$. As $\mathbf{S_L}$ is abelian, μ extends to $\mathrm{I}_{\mathbf{N_L}\rtimes\langle c\rangle}(\mu) = \mathbf{S_L} \rtimes \langle ab, bc\rangle$. Thus $\mathrm{Deg}\big(\mathbf{N_L} \rtimes \langle c\rangle \mid \mathbf{S_L}\big) = \{2^4, 4\}$.

Now $\mathrm{Deg}\big((\mathbf{N_L} \rtimes \langle c\rangle)/\mathbf{S_L}\big) = \mathrm{Deg}\big(\mathbf{Q} \times \langle c\rangle\big) = \{1^8, 2^2\}$. So

$$\mathrm{Deg}\big(\mathbf{N_L} \rtimes \langle c\rangle\big) = \{1^8, 2^6, 4\}. \tag{3.10.1}$$

Comparing (3.10.1) with (3.4.1), we see that

$$\mathrm{Inv}(\mathbf{S_L} \rtimes \mathbf{Q}) = \{1^4, 2^3\}, \qquad \mathrm{InvDef}_3(\mathbf{W} \rtimes \mathbf{Q} \bmod \mathbf{F}) = \{6^7\}. \tag{3.10.2}$$

Let χ be one of the two irreducible characters of $\mathbf{F}$ of degree 9. Both a and c invert $\mathrm{Z}(\mathbf{F})$. So ac centralizes $\mathrm{Z}(\mathbf{F})$. Hence $\mathrm{I}_{\mathbf{N_L}\rtimes\langle c\rangle}(\chi) = \mathbf{S_L} \rtimes \langle b, ac\rangle \cong \mathbf{S_L} \rtimes \mathbf{Q} = \mathbf{N_L}$. Using (3.4.1), we see that $\mathrm{Deg}\big(\mathrm{I}_{\mathbf{N_L}\rtimes\langle c\rangle}(\chi)\big) = \mathrm{Deg}(\mathbf{N_L}) = \{1^4, 2^5\}$.

Since χ extends to $\mathbf{W}$, it also extends to $\mathrm{I}_{\mathbf{F}\rtimes(\mathbf{N_L}\rtimes\langle c\rangle)}(\chi)$. Then from Clifford theory

$$\mathrm{Deg}\big(\mathbf{W} \rtimes (\mathbf{Q} \times \langle c\rangle) \mid \mathrm{Z}(\mathbf{F})\big) = \{18^4, 36^5\}. \tag{3.10.3}$$

Comparing (3.10.3) with (3.4.4), we see that

$$\mathrm{Inv}\big(\mathbf{W} \rtimes \mathbf{Q} \mid \mathrm{Z}(\mathbf{F})\big) = \{18^2, 36^2\}, \qquad \mathrm{InvDef}_3\big(\mathbf{W} \rtimes \mathbf{Q} \mid \mathrm{Z}(\mathbf{F})\big) = \{4^4\}. \tag{3.10.4}$$

If $x \in \mathbf{Q} \times \langle c\rangle$ has order 2, then $\mathbf{Q}$ centralizes $\langle x\rangle$ and hence stabilizes $\mathrm{C}_{\overline{\mathbf{F}}^*}(x)$. By Proposition 3.1.4 and Lemma 3.1.7, the group $\mathbf{Q}$ acts regularly on each of its nontrivial orbits in $\overline{\mathbf{F}}^*$. So $|\mathbf{Q}| \mid |\mathrm{C}_{\overline{\mathbf{F}}^*}(x)| - 1$. Hence $|\mathrm{C}_{\overline{\mathbf{F}}^*}(x)| = 1$ or 9. We conclude that the nontrivial elements of $\mathrm{C}_{\overline{\mathbf{F}}^*}(x)$ form either a single $\mathbf{Q}$-orbit or the empty set.

From (3.4.2) we know that $\mathrm{Orb}\big(\mathbf{N_L}, (\overline{\mathbf{F}}^*)^{\#}\big) = \{8, 24^3\}$. The $\mathbf{N_L}$-orbit of length 8 must be an $\mathbf{N_L} \times \langle c\rangle$-orbit, while one of the three $\mathbf{N_L}$-orbits of length 24 must be an $\mathbf{N_L} \times \langle c\rangle$-orbit, and the other two either join to form a single $\mathbf{N_L} \times \langle c\rangle$-orbit of length 48, or are both $\mathbf{N_L} \times \langle c\rangle$-orbits. If $\nu \in \overline{\mathbf{F}}^*$ lies in an $\mathbf{N_L} \rtimes \langle c\rangle$-orbit of length 24, then $\mathrm{I}_{\mathbf{N_L} \rtimes \langle c\rangle}(\nu)$ is isomorphic to $\mathbb{Z}_2$. Thus some $\mathbf{N_L} \rtimes \langle c\rangle$-conjugate of $\mathrm{I}_{\mathbf{N_L} \rtimes \langle c\rangle}(\nu)$ is contained in the Sylow 2-subgroup $\mathbf{Q} \times \langle c\rangle$ of $\mathbf{N_L} \rtimes \langle c\rangle$. Since τ inverts $\overline{\mathbf{F}}^*$, it follows that $\mathrm{I}_{\mathbf{N_L} \rtimes \langle c\rangle}(\nu)$ is conjugate to one of the remaining two subgroups of $\mathbf{Q} \times \langle c\rangle$ of order 2. By the previous paragraph, each of these $\mathbb{Z}_2$'s centralizes elements from at most one $\mathbf{N_L} \rtimes \langle c\rangle$-orbit of $(\overline{\mathbf{F}}^*)^{\#}$. Hence there are at most two $\mathbf{N_L} \rtimes \langle c\rangle$-orbits of length 24. But then there is exactly one such orbit, coinciding with one of the $\mathbf{N_L}$-orbits of length 24. The remaining two $\mathbf{N_L}$-orbits of length 24 must fuse in $\mathbf{N_L} \rtimes \langle c\rangle$. Thus

$$\mathrm{Orb}\big(\mathbf{N_L} \rtimes \langle c\rangle, (\overline{\mathbf{F}}^*)^{\#}\big) = \{8, 24, 48\}.$$

Let ν_1 lie in the $\mathbf{N_L} \rtimes \langle c\rangle$-orbit of length 8. Without loss of generality $\mathbf{S_L} \leq \mathrm{I}_{\mathbf{N_L} \rtimes \langle c\rangle}(\nu_1)$. We know there exists an involution $x \in \{\tau, c, \tau c\}$ such that $\mathrm{I}_{\mathbf{N_L} \rtimes \langle c\rangle}(\nu_1) = \mathbf{S_L} \rtimes \langle x\rangle$. We cannot have $x \neq \tau$, since τ inverts $\overline{\mathbf{F}}^*$. So x is either c or τc. In both cases x inverts $\mathbf{S_L}$, because τ centralizes $\mathbf{S_L}$ and c inverts $\mathbf{S_L}$. Hence $\mathrm{I}_{\mathbf{N_L} \rtimes \langle c\rangle}(\nu_1) = \mathbf{S_L} \rtimes \langle x\rangle \cong \mathfrak{S}_3$. Invoking Theorems 1.2.15 and 1.2.16 we see that

$$\mathrm{Deg}\big(\overline{\mathbf{F}} \rtimes (\mathbf{N_L} \rtimes \langle c\rangle) \mid \nu_1\big) = \{8^2, 16\}. \tag{3.10.5}$$

Let $\nu_2 \in (\overline{\mathbf{F}}^*)^\#$ have $\mathbf{N_L} \rtimes \langle c\rangle$-orbit length 24. Then $\mathrm{I}_{\mathbf{N_L}\rtimes\langle c\rangle}(\nu_2) \cong \mathbb{Z}_2$. So

$$(3.10.6) \qquad \operatorname{Deg}\big(\overline{\mathbf{F}} \rtimes (\mathbf{N_L} \rtimes \langle c\rangle) \mid \nu_2\big) = \{24^2\}.$$

Let $\nu_3 \in (\overline{\mathbf{F}}^*)^\#$ have $\mathbf{N_L} \rtimes \langle c\rangle$-orbit length 48. Then ν_3 induces an irreducible character of $\overline{\mathbf{F}} \rtimes \big(\mathbf{N_L} \rtimes \langle c\rangle\big)$ of degree 48. So

$$(3.10.7) \qquad \operatorname{Deg}\big(\overline{\mathbf{F}} \rtimes (\mathbf{N_L} \rtimes \langle c\rangle) \mid \nu_3\big) = \{48\}.$$

From (3.10.5), (3.10.6) and (3.10.7) we obtain

$$(3.10.8) \qquad \operatorname{Deg}\big(\mathbf{W} \rtimes (\mathbf{Q} \times \langle c\rangle) \mid \mathbf{F}/\,\mathrm{Z}(\mathbf{F})\big) = \{8^2, 16, 24^2, 48\}.$$

Comparing (3.10.8) with (3.4.3) we conclude that

$$(3.10.9) \quad \operatorname{Inv}\big(\mathbf{W} \rtimes \mathbf{Q} \mid \mathbf{F}/\,\mathrm{Z}(\mathbf{F})\big) = \{8, 24\}, \qquad \operatorname{InvDef}_3\big(\mathbf{W} \rtimes \mathbf{Q} \mid \mathbf{F}/\,\mathrm{Z}(\mathbf{F})\big) = \{6, 5\}.$$

From (3.10.2), (3.10.4) and (3.10.8) we have

$$(3.10.10) \qquad \begin{aligned} \operatorname{Inv}(\mathbf{W} \rtimes \mathbf{Q}) &= \{1^4, 2^3, 8, 18^2, 24, 36^2\}, \\ \operatorname{InvDef}_3(\mathbf{W} \rtimes \mathbf{Q}) &= \{6^8, 5, 4^4\}. \end{aligned}$$

We now can prove the following proposition.

Proposition 3.10.11. *The group $\mathrm{N}_{\mathbf{M_c}}(C_3) = \mathbf{W} \rtimes \mathbf{Q}$ has unique 3-block, which necessarily induces the principal 3-block, B_0, of $\mathbf{M_c}$. Hence*

$$(3.10.12) \qquad \begin{aligned} &\mathrm{k}(\,C_3, \mathsf{B}_0, 6, \overline{\mathbf{M_c}.2}\,) = 8, && \mathrm{k}(\,C_3, \mathsf{B}_0, 5, \overline{\mathbf{M_c}.2}\,) = 1, \\ &\mathrm{k}(\,C_3, \mathsf{B}_0, 4, \overline{\mathbf{M_c}.2}\,) = 4, && \mathrm{k}(\,C_3, \mathsf{B}_0, 3, \overline{\mathbf{M_c}.2}\,) = 0, \\ &\mathrm{k}(\,C_3, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = 0, && \textit{for all other values of } d. \end{aligned}$$

PROOF. This follows immediately from Proposition 3.4.6 and (3.10.10). $\square$

From Table 3.1 on page 48 and Lemma 3.7.1, the normalizer in $\mathbf{M_c}.2$ of each of the chains C_3, C_5 and C_6 coincides with $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{W}) = \mathbf{W} \rtimes (\langle d\mathbf{Q}\rangle)$. Hence $\mathrm{k}(C_3, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}) = \mathrm{k}(C_5, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}) = \mathrm{k}(C_6, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2})$ for all values of d.

3.11. The Invariant Conjecture for the prime $p = 3$

We obtain the invariant characters of the principal block B_0 of $\mathbf{M_c}$ from [**Con85**, p100]. We list here these characters and their defects:

Character	χ_1	χ_2	χ_3	χ_4	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{15}	χ_{20}
Degree	1	22	231	252	896	896	1750	3520	3520	4500	4752	5544	9625
3-Defect	6	6	5	4	6	6	6	6	6	4	3	4	6

Thus

$$\begin{aligned} &\mathrm{k}(C_1, \mathsf{B}_0, 6, \overline{\mathbf{M_c}.2}) = 8, && \mathrm{k}(C_1, \mathsf{B}_0, 5, \overline{\mathbf{M_c}.2}) = 1, \\ &\mathrm{k}(C_1, \mathsf{B}_0, 4, \overline{\mathbf{M_c}.2}) = 3, && \mathrm{k}(C_1, \mathsf{B}_0, 3, \overline{\mathbf{M_c}.2}) = 1, \\ &\mathrm{k}(C_1, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}) = 0, && \text{for all other values of } d. \end{aligned} \tag{3.11.1}$$

We can now prove the following

THEOREM 3.11.2. *The Invariant Conjecture holds for the McLaughlin simple group and the prime $p = 3$.*

PROOF. From Conjecture 1.4.4 and Table 3.1 on page 48, we must prove

$$\begin{aligned}
&\mathrm{k}(\,C_1, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) + \mathrm{k}(\,C_3, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = \\
&\qquad\qquad \mathrm{k}(\,C_2, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) + \mathrm{k}(\,C_4, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,),
\end{aligned} \tag{3.11.3}$$

for all values of $d \in \mathbb{Z}$.

From (3.8.8), (3.9.9), (3.10.12) and (3.11.1) we obtain the following sums for the equation above for $d \in \{3, 4, 5, 6\}$:

3-Defect	C_1		C_3		C_2		C_4
6	8	+	8	=	8	+	8
5	1	+	1	=	1	+	1
4	3	+	4	=	3	+	4
3	1	+	0	=	0	+	1

TABLE 3.3. The Invariant Conjecture for $p = 3$

The summands in (3.11.3) are zero for all other values of d. This completes the proof of the theorem. ☐

3.12. The Schur Multiplier of $\mathbf{M_c}$

We let $\mathbf{A}.\mathbf{M_c}$ denote the universal covering group of $\mathbf{M_c}$ by its Schur Multiplier $\mathbf{A}$. In [**Gr87**], R. Greiss shows that $\mathbf{A}$ is isomorphic to $\mathbb{Z}_3$. The arguments he uses are somewhat terse. So we expound them in more detail here. We will assume only that $\mathbf{A}$ has order a non-trivial power of 3.

In this section $\mathbf{A}_1$ will denote a fixed subgroup of $\mathbf{A}$, properly contained in $\mathbf{A}$. We let $\widehat{\mathbf{M}_\mathbf{c}} = \mathbf{A}^1.\mathbf{M}_\mathbf{c}$ denote the factor extension $\mathbf{A}.\mathbf{M}_\mathbf{c}/\mathbf{A}_1$ of $\mathbf{M}_\mathbf{c}$ by $\mathbf{A}^1 = \mathbf{A}/\mathbf{A}_1$. So $\widehat{\mathbf{M}_\mathbf{c}}$ is a perfect central extension of $\mathbf{M}_\mathbf{c}$ by a non-trivial 3-group, and any factor extension $\widehat{\mathbf{M}_\mathbf{c}}/\mathbf{A}^1_1$ of $\mathbf{M}_\mathbf{c}$ by $\mathbf{A}^1/\mathbf{A}^1_1$ is non-split, for every subgroup $\mathbf{A}^1_1 < \mathbf{A}^1$.

If X is any subgroup of $\mathbf{M}_\mathbf{c}$ then we will use $\mathbf{A}^1.\mathrm{X}$ or $\widehat{\mathrm{X}}$ to denote its inverse image in $\widehat{\mathbf{M}_\mathbf{c}}$. We note that there is a natural conjugation action of $\mathbf{M}_\mathbf{c}$ on $\widehat{\mathbf{M}_\mathbf{c}}$, since $\widehat{\mathbf{M}_\mathbf{c}}$ is a central extension of $\mathbf{M}_\mathbf{c}$.

We will use the notation of Section 3.1. In particular, $\mathbf{F}$ is an extra-special 3-group of $\mathbf{M}_\mathbf{c}$ of type 3^{1+4}_+, and $\mathrm{N}_{\mathbf{M}_\mathbf{c}}(\mathbf{F}) = \mathbf{F} \rtimes \mathbf{L}$, where $\mathbf{L} \cong \mathrm{SL}(2,5).2$. We let $\mathbf{C} = \mathbf{L}'$. So $\mathbf{C} \cong \mathrm{SL}(2,5)$. Also $\mathbf{W}$ will denote a fixed Sylow 3-subgroup of $\mathbf{M}_\mathbf{c}$ containing $\mathbf{F}$, and $\mathbf{S}_\mathbf{L}$ will denote the intersection of $\mathbf{W}$ with $\mathbf{L}$. Since $\mathbf{F} = \mathrm{O}_3\big(\mathbf{F} \rtimes \mathbf{L}\big)$, it follows that $\hat{\mathbf{F}} = \mathrm{O}_3\big(\mathbf{A}^1.(\mathbf{F} \rtimes \mathbf{L})\big)$.

Lemma 3.12.1. *$\widehat{\mathbf{L}}$ splits over $\mathbf{A}^1$.*

Proof. From the Atlas, the group $\mathbf{C} \cong 2.\mathfrak{A}_5$ is isomorphic to the universal covering group of $\mathfrak{A}_5$. Hence $\widehat{\mathbf{C}}$ splits over $\mathbf{A}^1$. But $\mathbf{C}$ contains a Sylow 3-subgroup $\mathbf{S}_\mathbf{L}$ of $\mathbf{L}$. So $\widehat{\mathbf{S}_\mathbf{L}}$ splits over $\mathbf{A}^1$. We conclude that $\widehat{\mathbf{L}}$ splits over $\mathbf{A}^1$. $\square$

We will denote by $\mathbf{L}^1$ a fixed complement to $\mathbf{A}^1$ in $\widehat{\mathbf{L}}$. So $\mathbf{L}^1$ is a complement to $\hat{\mathbf{F}}$ in $\mathbf{A}^1.\big(\mathbf{F} \rtimes \mathbf{L}\big)$.

Lemma 3.12.2. *$\hat{\mathbf{F}}$ does not split over $\mathbf{A}^1$.*

Proof. Suppose there exists a complement $\mathbf{F}^1$ to $\mathbf{A}^1$ in $\hat{\mathbf{F}}$. Since $\mathbf{A}^1$ is central, there is a one-to-one correspondence between the complements to $\mathbf{A}^1$ in $\hat{\mathbf{F}}$ and the elements of the set $\mathrm{H}^1(\mathbf{F}^1, \mathbf{A}^1) = \mathrm{Hom}(\mathbf{F}^1, \mathbf{A}^1) = \mathrm{Hom}(\overline{\mathbf{F}}, \mathbf{A}^1)$. The complement corresponding to $\mathrm{f} \in \mathrm{Hom}(\mathbf{F}^1, \mathbf{A}^1)$ is the set $\{\mathrm{f}(x) \times x \mid x \in \mathbf{F}^1\}$.

Now $|\mathrm{Hom}(\overline{\mathbf{F}}, \mathbf{A}^1)| = |\Omega(\mathbf{A}^1)|^4$ is a power of 3, as $\overline{\mathbf{F}}$ is an elementary abelian 3-group. Consider the central involution τ^1 of $\mathbf{L}^1$. This element normalizes $\hat{\mathbf{F}}$ and $\mathbf{A}^1$. Thus it permutes the complements to $\mathbf{A}^1$ in $\hat{\mathbf{F}}$. Since there are an odd number of complements, there is at least one which is fixed by τ^1. We assume that τ^1 fixes $\mathbf{F}^1$. As τ^1 inverts $\overline{\mathbf{F}}$ and centralizes $\mathbf{A}^1$, it inverts all homomorphisms $\overline{\mathbf{F}} \to \mathbf{A}^1$. So τ^1 fixes no non-trivial elements of the 3-group $\mathrm{Hom}(\overline{\mathbf{F}}, \mathbf{A}^1)$. Hence $\mathbf{F}^1$ is the unique complement fixed by τ^1.

Let x be an arbitrary element of $\mathbf{L}^1 \leq \mathrm{C}_{\widehat{\mathbf{M}}_{\mathbf{c}}}(\tau^1)$. Then $(\mathbf{F}^1)^x$ is a complement to $\mathbf{A}^1$ in $\hat{\mathbf{F}}$ and $((\mathbf{F}^1)^x)^{\tau^1} = (\mathbf{F}^1)^{x\tau^1} = (\mathbf{F}^1)^{\tau^1 x} = (\mathbf{F}^1)^x$. As $\mathbf{F}^1$ is the unique complement fixed by τ^1, this implies that $(\mathbf{F}^1)^x = \mathbf{F}^1$. So $\mathbf{L}^1$ normalizes $\mathbf{F}^1$. Hence $\mathbf{F}^1 \rtimes \mathbf{L}^1$ is a complement to $\mathbf{A}^1$ in $\mathbf{A}^1.(\mathbf{F} \rtimes \mathbf{L})$. Since $\widehat{\mathbf{W}}$ is a subgroup of $\mathbf{A}^1.(\mathbf{F} \rtimes \mathbf{L})$, it splits over $\mathbf{A}^1$. This implies that $\widehat{\mathbf{M}}_{\mathbf{c}}$ itself splits over $\mathbf{A}^1$, as $\widehat{\mathbf{W}}$ is a Sylow 3-subgroup of $\widehat{\mathbf{M}}_{\mathbf{c}}$. This is impossible. The lemma now follows. □

We now prove

LEMMA 3.12.3. $\hat{\mathbf{F}}' = \widehat{\mathbf{F}'} = \widehat{\mathrm{Z}(\mathbf{F})}$.

PROOF. Since $\mathrm{Z}(\mathbf{F})$ is cyclic, $\widehat{\mathrm{Z}(\mathbf{F})}$ is abelian. The derived group $\hat{\mathbf{F}}'$ of $\hat{\mathbf{F}}$ is contained in $\widehat{\mathrm{Z}(\mathbf{F})}$, since $\mathbf{F}' = \mathrm{Z}(\mathbf{F})$. Furthermore, $\hat{\mathbf{F}}'$ is stabilized by $\langle a \rangle \in \mathbf{L}$. Let $\mathbf{A}^1_1 = \mathrm{C}_{\hat{\mathbf{F}}'}(a)$ and $\mathrm{Z}^1 = [\hat{\mathbf{F}}', a]$. Then by [**As86**, 24.6] we have $\hat{\mathbf{F}}' = \mathbf{A}^1_1 \times \mathrm{Z}^1$. Also $\mathbf{A}^1_1$ is contained in $\mathbf{A}^1$ and $\widehat{\mathrm{Z}(\mathbf{F})} = \mathbf{A}^1 \times \mathrm{Z}^1$.

We claim that $\mathbf{A}^1_1 = \mathbf{A}^1$. If not, $\mathbf{A}^1/\mathbf{A}^1_1$ is a non-trivial quotient of $\mathbf{A}$. Since $\mathbf{A}^1$ is an arbitrary non-trivial quotient of $\mathbf{A}$, we can replace it by $\mathbf{A}^1/\mathbf{A}^1_1$. So for the rest of the proof we assume we have done so. This is equivalent to assuming that $\mathbf{A}^1_1 = \{1\}$.

By [**As86**, 24.6], the abelian group $\hat{\mathbf{F}}/\hat{\mathbf{F}}'$ can be written as $\hat{\mathbf{F}}/\hat{\mathbf{F}}' = \mathrm{C}_{\hat{\mathbf{F}}/\hat{\mathbf{F}}'}(\tau) \times [\hat{\mathbf{F}}/\hat{\mathbf{F}}', \tau]$. Since τ centralizes $\mathbf{A}^1\hat{\mathbf{F}}'/\hat{\mathbf{F}}'$ but inverts $\hat{\mathbf{F}}/(\mathbf{A}^1\hat{\mathbf{F}}') = \hat{\mathbf{F}}/\widehat{\mathrm{Z}(\mathbf{F})} \cong \overline{\mathbf{F}}$, we have

$$\hat{\mathbf{F}}/\hat{\mathbf{F}}' = \left(\mathbf{A}^1\hat{\mathbf{F}}'/\hat{\mathbf{F}}'\right) \times [\hat{\mathbf{F}}/\hat{\mathbf{F}}', \tau].$$

Let $\mathbf{F}^1$ be the inverse image of $[\hat{\mathbf{F}}/\hat{\mathbf{F}}', \tau]$ in $\hat{\mathbf{F}}$. Then $\mathbf{A}^1\hat{\mathbf{F}}' \cap \mathbf{F}^1 = \hat{\mathbf{F}}'$, since $\left(\mathbf{A}^1\hat{\mathbf{F}}'/\hat{\mathbf{F}}'\right) \cap \left(\mathbf{F}^1/\hat{\mathbf{F}}'\right) = \{1\}$. But $\mathbf{A}^1 \cap \hat{\mathbf{F}}' = \mathbf{A}^1_1 = \{1\}$. So $\mathbf{A}^1 \cap \mathbf{F}^1 = \{1\}$. It follows that $\mathbf{F}^1$ is a complement to $\mathbf{A}^1$ in $\hat{\mathbf{F}}$, contradicting Lemma 3.12.2. This proves our claim. The lemma follows. □

COROLLARY 3.12.4. *The* 3*-group* $\hat{\mathbf{F}}$ *is of class two. Hence* $\hat{\mathbf{F}}' = \mathrm{Z}(\hat{\mathbf{F}})$*, while both* $\hat{\mathbf{F}}'$ *and* $\mathbf{A}^1$ *are elementary abelian.*

PROOF. Since $[\mathbf{F}, \mathbf{F}'] = \{1\}$, we have $[\hat{\mathbf{F}}, \hat{\mathbf{F}}'] \leq \mathbf{A}^1$. So commutation in $\hat{\mathbf{F}}$ induces a $\langle\tau\rangle$-invariant bilinear map $\hat{\mathbf{F}}/\hat{\mathbf{F}}' \times \hat{\mathbf{F}}'/\mathbf{A}^1 \to \mathbf{A}^1$. Since $\hat{\mathbf{F}}'/\mathbf{A}^1$ and $\mathbf{A}^1$ are centralized by τ, while $\hat{\mathbf{F}}/\hat{\mathbf{F}}'$ is inverted by τ, the above map is trivial. Hence $[\hat{\mathbf{F}}, \hat{\mathbf{F}}'] = \{1\}$.

As $\hat{\mathbf{F}}/\hat{\mathbf{F}}' \cong \overline{\mathbf{F}}$ is elementary abelian, [**R93**, 5.2.5] implies that $\hat{\mathbf{F}}'/[\hat{\mathbf{F}}, \hat{\mathbf{F}}'] = \hat{\mathbf{F}}'$ is also elementary abelian. In particular $\mathbf{A}^1$ is elementary abelian. □

COROLLARY 3.12.5. $\mathbf{C} = \mathbf{L}'$ *centralizes* $\hat{\mathbf{F}}'$*. The elementary abelian* 3*-group* $\hat{\mathbf{F}}' = \mathrm{Z}(\hat{\mathbf{F}})$ *can be written as*

$$\hat{\mathbf{F}}' = \mathbf{A}^1 \times \mathrm{Z}^1,$$

where $\mathrm{Z}^1 \cong \mathbb{Z}_3$ *is an* $\mathbf{L}$*-invariant subgroup of* $\hat{\mathbf{F}}'$*.*

PROOF. The action of $\mathbf{M_c}$ on $\widehat{\mathbf{M}}_\mathbf{c}$ restricts to an action of $\mathbf{C}$ and $\mathbf{L}$ on the 3-group $\hat{\mathbf{F}}' = \mathrm{Z}(\hat{\mathbf{F}}) = \widehat{\mathrm{Z}(\mathbf{F})}$. By Lemma 3.1.3, the action of $\mathbf{C}$ on each section for the normal series $\hat{\mathbf{F}}' > \mathbf{A}^1 > \{1\}$ of $\hat{\mathbf{F}}'$ is trivial. Since $\mathbf{C} \cong \mathrm{SL}(2,5)$ has no non-trivial quotients which are 3-groups, this implies that it centralizes $\hat{\mathbf{F}}'$.

Recall there exists an element a of order 4 in $\mathbf{L}\backslash\mathbf{C}$ which inverts $\mathrm{Z}(\mathbf{F})$. By [**As86**, 24.6] we may write $\hat{\mathbf{F}}' = \mathrm{Z}(\hat{\mathbf{F}}) = \mathrm{C}_{\hat{\mathbf{F}}'}(a) \times [\hat{\mathbf{F}}', a]$. But a centralizes $\mathbf{A}^1$ and inverts $\hat{\mathbf{F}}'/\mathbf{A}^1$. So $\hat{\mathbf{F}}' = \mathbf{A}^1 \times \mathrm{Z}^1$, where $\mathrm{Z}^1 = [\hat{\mathbf{F}}', a]$. This decomposition is $\mathbf{L}$-invariant as $\mathbf{L} = \langle \mathbf{C}, a \rangle$, and $\mathbf{C}$ centralizes $\hat{\mathbf{F}}'$. □

For the remainder of this chapter we let $\mathbf{S}_{\mathbf{L}^1}$ denote the Sylow 3-subgroup of $\mathbf{L}^1$ contained in $\widehat{\mathbf{W}}$. So $\mathbf{S}_{\mathbf{L}^1}$ is cyclic of order 3.

COROLLARY 3.12.6. $\mathrm{Z}(\widehat{\mathbf{W}}) = \mathrm{Z}(\hat{\mathbf{F}})$.

PROOF. Since $\mathrm{Z}(\mathbf{W}) = \mathrm{Z}(\mathbf{F})$, we have $\mathrm{Z}(\widehat{\mathbf{W}}) \leq \widehat{\mathrm{Z}(\mathbf{F})} = \mathrm{Z}(\hat{\mathbf{F}})$. By the previous corollary $[\mathbf{C}, \mathrm{Z}(\hat{\mathbf{F}})] = 1$. So the Sylow 3-subgroup $\mathbf{S}_{\mathbf{L}^1} = \widehat{\mathbf{W}} \cap \mathbf{L}^1$ of $\mathbf{L}^1$ centralizes $\mathrm{Z}(\hat{\mathbf{F}})$. But $\mathbf{S}_{\mathbf{L}^1}\hat{\mathbf{F}} = \widehat{\mathbf{W}}$. The result now follows immediately. □

The following lemma could have been proved earlier. We use it here to show that $\mathbf{A}$ has order 3.

LEMMA 3.12.7. $\mathbf{C}$ *acts irreducibly, but not absolutely irreducibly on* $\overline{\mathbf{F}}$. *Also* $\mathbf{L}$ *acts absolutely irreducibly on* $\overline{\mathbf{F}}$.

PROOF. From Proposition 3.1.14 the group $\mathbf{L}$ acts transitively on $\overline{\mathbf{F}}^{\#}$, and the stabilizer of a point is isomorphic to $\mathbf{S_L}$. Since $\mathbf{S_L} \leq \mathbf{C}$ has index 40 in $\mathbf{C}$ it follows that $\mathrm{Orb}(\mathbf{C} \mid \overline{\mathbf{F}}^{\#}) = \{40^2\}$. Hence $\mathbf{C}$ acts irreducibly on $\overline{\mathbf{F}}$. Consider $\overline{\mathbf{F}}$ as an additive $\mathrm{GF}(3)\mathbf{C}$-module. Then $\mathfrak{F} = \mathrm{End}_{\mathrm{GF}(3)\mathbf{C}}(\overline{\mathbf{F}})$ is a finite field of characteristic three.

Recall from Lemma 3.7.2 that $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{F}) = \mathbf{F} \rtimes (\langle d \rangle \mathbf{L})$, where d is an element of $\mathbf{M_c}.2$ of order four, $[d, \mathbf{C}] = 1$, and $[d, a] = \tau$.

We identify $\mathrm{GL}(4,3)$ with $\mathrm{End}_{\mathrm{GF}(3)}(\overline{\mathbf{F}})$. Then we can regard $\langle d \rangle \mathbf{L}$ as a subgroup of $\mathrm{GL}(4,3)$, as $\langle d \rangle \mathbf{L}$ acts faithfully on $\overline{\mathbf{F}}$. Thus $d \in \mathfrak{F}$, and hence $4 \bigm| |\mathfrak{F}^{\#}|$. Since $\mathfrak{F}^{\#} \leq \mathrm{GL}(4,3)$, this implies that $\mathfrak{F} \cong \mathrm{GF}(9)$ or $\mathrm{GF}(81)$. However $\mathrm{GF}(81)^{\#}$ is a self centralizing subgroup of $\mathrm{GL}(4,3)$, and thus does not contain $\mathbf{C}$. So $\mathfrak{F} \cong \mathrm{GF}(9)$. This proves the first statement.

Now $\mathbf{C} \leq \mathbf{L}$. So $\mathrm{End}_{\mathrm{GF}(3)\mathbf{L}}(\overline{\mathbf{F}}) \subseteq \mathfrak{F}$. But $[\mathbf{L}, d] \neq 1$. So $\mathrm{End}_{\mathrm{GF}(3)\mathbf{L}}(\overline{\mathbf{F}}) < \mathfrak{F}$. We conclude that $\mathrm{End}_{\mathrm{GF}(3)\mathbf{L}}(\overline{\mathbf{F}}) = \mathrm{GF}(3)$. This proves the second statement. $\square$

We now specialize the previous results to $\mathbf{A}^1 = \mathbf{A}$. Then $\widehat{\mathbf{M}_\mathbf{c}}$ becomes the universal covering group $\mathbf{A}.\mathbf{M_c}$ of $\mathbf{M_c}$ by its Schur multiplier $\mathbf{A}$, and $\widehat{\mathrm{X}}$ becomes $\mathbf{A}.\mathrm{X}$, for any $\mathrm{X} \leq \mathbf{M_c}$.

Theorem 3.12.8. $\mathbf{A} \cong \mathbb{Z}_3$.

Proof. By Corollary 3.12.4, the group $\mathbf{A}$ is an elementary abelian 3-group. Suppose $\mathbf{A} \cong \mathbb{Z}_3^n$, for some $n \in \mathbb{Z}$. We can find $x_1, \ldots, x_n, y_1, \ldots, y_n \in \hat{\mathbf{F}}$ such that $\big\{[x_i, y_i] \mid i = 1, \ldots, n\big\}$ is a basis for $\mathbf{A}$.

Commutation in $\hat{\mathbf{F}}$ gives rise to the $\mathbf{L}$-invariant alternating bilinear forms $\rho_i : \overline{\mathbf{F}} \times \overline{\mathbf{F}} \to \mathrm{GF}(3)$, for $i = 1, \ldots, n$, where $\rho_i(\overline{x}, \overline{y})$ is defined by the equation

$$[x, y] = \prod_{i=1}^{n} [x_i, y_i]^{\rho_i(\overline{x}, \overline{y})},$$

for any $x, y \in \hat{\mathbf{F}}$ with respective images $\overline{x}, \overline{y} \in \overline{\mathbf{F}}$. Each ρ_i induces a map $f_i \in \mathrm{Hom}_{\mathrm{GF(3)}\mathbf{L}}(\overline{\mathbf{F}}, \overline{\mathbf{F}}^*)$, given by $f_i(\overline{x}) : \overline{y} \to \rho_i(\overline{x}, \overline{y})$, for all $\overline{x}, \overline{y} \in \overline{\mathbf{F}}$. The map f_i is non-trivial, as $\rho_i(\overline{x}_i, \overline{y}_i) = 1_{\mathrm{GF(3)}}$.

Since $\overline{\mathbf{F}}$ and $\overline{\mathbf{F}}^*$ are irreducible $\mathbf{L}$-modules, Schur's Lemma implies that f_i is in fact an $\mathbf{L}$-isomorphism of $\overline{\mathbf{F}}$ onto $\overline{\mathbf{F}}^*$, for $i = 1, \ldots, n$. Thus each $f_1^{-1} f_i$ is an element of $\mathrm{End}_{\mathrm{GF(3)}\mathbf{L}}(\overline{\mathbf{F}})$. But $\mathbf{L}$ acts absolutely irreducibly on $\overline{\mathbf{F}}$ by Lemma 3.12.7. So $f_1^{-1} f_i = \lambda_i$, for some $\lambda_i \in \mathrm{GF(3)}^{\#}$.

Suppose $n > 1$. Then

$$0 = \rho_2(\overline{x}_1, \overline{y}_1) = (f_2(\overline{x}_1))(\overline{y}_1) = (\lambda_2 f_1(\overline{x}_1))(\overline{y}_1) = \lambda_2\, \rho_1(\overline{x}_1, \overline{y}_1) = \lambda_2 \neq 0.$$

This contradiction forces $n = 1$. We conclude that $\mathbf{A} \cong \mathbb{Z}_3$. ☐

3.13. The Character Degrees of $\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M})$

The action of $\mathbf{M_c}$ on $\widehat{\mathbf{M_c}}$ restricts to an action of $\mathbf{M} \cong \widehat{\mathbf{M}}/\mathbf{A}$ on $\widehat{\mathbf{E}}$. If $x \in \mathbf{M}$, then $\tilde{y}^x = \tilde{y}^{\tilde{x}}$, for any $\tilde{y} \in \widehat{\mathbf{E}}$, where $\tilde{x}$ is any element of $\widehat{\mathbf{M}}$ lying over x. Notice that y^x is the image of $\tilde{y}^x$ in $\mathbf{M_c}$, where y is the image of $\tilde{y}$ in $\mathbf{E}$.

Recall from (3.5.2) the chain $\mathrm{K} < \mathrm{A}_4 < \mathrm{S}_4 < \mathrm{A}_6 < \mathbf{M}$ of subgroups of $\mathbf{M}$, where $\mathrm{A}_6 = \mathbf{M}' \cong \mathfrak{A}_6$, $\mathrm{S}_4 \cong \mathfrak{S}_4$, $\mathrm{A}_4 = \mathrm{S}_4{}' \cong \mathfrak{A}_4$ and $\mathrm{K} = \mathrm{O}_2(\mathrm{A}_4) \cong \mathbb{Z}_2^2$.

PROPOSITION 3.13.1. $\widehat{\mathbf{E}} \cong \mathbb{Z}_3^5$.

PROOF. From Lemma 3.5.1 the elementary abelian 3-group $\mathbf{E}$ is isomorphic to the regular GF(3)K-module. In particular $\mathbf{E} \cong \mathbf{E}_1 \oplus \mathbf{E}_2 \oplus \mathbf{E}_3 \oplus \mathbf{E}_4$ as GF(3)K-modules, where each $\mathbf{E}_i$ is cyclic of order 3.

Suppose $i \neq j$. Commutation in $\widehat{\mathbf{E}}$ gives rise to an K-invariant bilinear map $\widehat{\mathbf{E}}_i/\mathbf{A} \times \widehat{\mathbf{E}}_j/\mathbf{A} \to \mathbf{A}$. By the regularity of $\mathbf{E}$, we can find $x \in \mathrm{K}$ which centralizes one of $\mathbf{E}_i, \mathbf{E}_j$ and inverts the other. So the map must be trivial. Hence $\widehat{\mathbf{E}}_i$ commutes with $\widehat{\mathbf{E}}_i$. Moreover each $\widehat{\mathbf{E}}_i$ is abelian, since each $\mathbf{E}_i$ is cyclic. We conclude that $\widehat{\mathbf{E}}$ is abelian.

By Proposition 3.1.4 the centralizer of $\mathbf{Q}$ in $\mathbf{E} \leq \mathbf{W}$ is trivial. Hence

$$\widehat{\mathbf{E}} = \mathbf{A} \times [\mathbf{Q}, \widehat{\mathbf{E}}] \cong \mathbf{A} \times \mathbf{E},$$

using [**As86**, 24.6]. So $\widehat{\mathbf{E}}$ is elementary abelian of order 3^5. $\square$

We will use the following lemma to show that the $(3B)$ conjugacy class of $\mathbf{M_c}$ fuses in $\widehat{\mathbf{M}}_\mathbf{c}$.

LEMMA 3.13.2. *Let* X *be a special* p*-group of type* p^{2+4}*, i.e.* $\mathrm{X}' = \Phi(\mathrm{X}) = \mathrm{Z}(\mathrm{X})$ *is elementary abelian of order* p^2*, while* $\overline{\mathrm{X}} = \mathrm{X}/\mathrm{X}'$ *is elementary abelian of order* p^4*. Then there is some element* $x \in \mathrm{X}$ *such that* $\mathrm{X}' = [x, \mathrm{X}]$*.*

PROOF. Let $\overline{\mathrm{X}} = \mathrm{X}/\mathrm{X}'$. We can regard $\overline{\mathrm{X}}$ and X' as vector spaces over $\mathrm{GF}(p)$ with dimensions 4 and 2, respectively. Commutation in X gives a bilinear map $[\ ,\ ] : \overline{\mathrm{X}} \times \overline{\mathrm{X}} \to \mathrm{X}'$.

Let $\overline{x}_1 \in \overline{\mathrm{X}}$. If $[\overline{x}_1, \overline{\mathrm{X}}] = \mathrm{X}'$, we are done. Otherwise, there exists $\overline{x}_2 \in \overline{\mathrm{X}}$ such that $\overline{\mathrm{X}} = \langle \overline{x}_2 \rangle \oplus \langle \overline{x}_1 \rangle^\perp$, where $\langle \overline{x}_1 \rangle^\perp = \{\overline{x} \in \overline{\mathrm{X}} \mid [\overline{x}_1, \overline{x}] = 1\}$. If $[\overline{x}_2, \overline{\mathrm{X}}] = \mathrm{X}'$, we are done. Otherwise $\overline{\mathrm{X}} = \langle \overline{x}_1 \rangle \oplus \langle \overline{x}_2 \rangle^\perp$ and $\langle \overline{x}_1 \rangle^\perp \neq \langle \overline{x}_2 \rangle^\perp$, since $\overline{x}_1 \in \langle \overline{x}_1 \rangle^\perp \backslash \langle \overline{x}_2 \rangle^\perp$. Thus $\langle \overline{x}_1 \rangle^\perp \cap \langle \overline{x}_2 \rangle^\perp$ is a 2-dimensional subspace of $\overline{\mathrm{X}}$. Let $\overline{x}_3, \overline{x}_4$ be generators for $\langle \overline{x}_1 \rangle^\perp \cap \langle \overline{x}_2 \rangle^\perp$. By the bilinearity of the commutator, it is clear that $[\langle \overline{x}_1, \overline{x}_2 \rangle, \langle \overline{x}_3, \overline{x}_4 \rangle] = \{1\}$.

Since $\overline{\mathrm{X}} = \langle \overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4 \rangle$, the vector space X' is spanned by $\{[\overline{x}_i, \overline{x}_j]\}$, for $1 \leq i < j \leq 4$. But then $\mathrm{X}' = \langle [\overline{x}_1, \overline{x}_2], [\overline{x}_3, \overline{x}_4] \rangle$, since all the other terms in the spanning set are trivial.

Let $\overline{x} = \overline{x}_1 + \overline{x}_3$. Then $[\overline{x}, \overline{x}_2] = [\overline{x}_1, \overline{x}_2]$ and $[\overline{x}, \overline{x}_4] = [\overline{x}_3, \overline{x}_4]$. So $[\overline{x}, \overline{\mathrm{X}}] = \mathrm{X}'$. The lemma follows.

$\square$

We need the next result in order to investigate the action of $\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M})$ on $\widehat{\mathbf{E}}$.

LEMMA 3.13.3. *The* $(3A)$ *conjugacy class of* $\mathbf{M_c}$ *splits in* $\widehat{\mathbf{M}}_\mathbf{c}$*, while the* $(3B)$ *conjugacy class of* $\mathbf{M_c}$ *fuses in* $\widehat{\mathbf{M}}_\mathbf{c}$*.*

PROOF. Let $\tilde{\alpha}$ be an element of $\widehat{\mathbf{M}}_\mathbf{c}$ whose image in $\mathbf{M_c}$ is α. Then $\tilde{\alpha}$ is central in $\widehat{\mathbf{W}}$, by Corollary 3.12.6. The derived group $\mathbf{C}^1$ of $\mathbf{L}^1$ centralizes $\widehat{\mathrm{Z}(\mathbf{F})}$, by Corollary 3.12.5. Hence

$$\mathrm{C}_{\widehat{\mathbf{M}}_\mathbf{c}}(\tilde{\alpha}) = \hat{\mathbf{F}} \rtimes \mathbf{C}^1 = \mathbf{A}.\big(\mathrm{C}_{\mathbf{M_c}}(\alpha)\big).$$

In particular, the conjugacy class $(3A)$ of $\mathbf{M_c}$ containing α splits in $\widehat{\mathbf{M}}_\mathbf{c}$.

By Lemma 3.13.2, there exist $x, y \in \hat{\mathbf{F}} \backslash \hat{\mathbf{F}}'$ such that $\mathbf{A} = \langle [x, y] \rangle$. This implies that the conjugacy class of $x\mathbf{A}$ in $\mathbf{M_c} = (\widehat{\mathbf{M}}_\mathbf{c})/\mathbf{A}$ fuses in $\widehat{\mathbf{M}}_\mathbf{c}$. But by Proposition 3.1.14, the element $x\mathbf{A} \in \mathbf{F} \backslash \mathbf{F}'$ comes from the $(3B)$ conjugacy class of $\mathbf{M_c}$. So this class fuses in $\widehat{\mathbf{M}}_\mathbf{c}$. $\square$

We fix $\rho \in \mathrm{Irr}(\mathbf{A})^{\#}$ for the remainder of this chapter. In the next lemma we compute the degrees and 3-defects of the irreducible characters of $\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M})$ which lie over ρ.

LEMMA 3.13.4. $\mathrm{Deg}(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \rho) = \{36^4, 45^4, 90^3, 144\}$. *Hence*

$$\mathrm{Def}_3(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \rho) = \{5^{12}\} \tag{3.13.5}$$

PROOF. As before $\tilde{\alpha}$ is an element of $\widehat{\mathbf{M}_\mathbf{c}}$ whose image in $\mathbf{M}_\mathbf{c}$ is α. We deduce from Proposition 3.1.12 and Lemma 3.13.3 that $\tilde{\alpha}$ lies in an $\mathbf{M}$-orbit of size 20, and $\mathbf{M}$ has three such orbits in $\widehat{\mathbf{E}}$.

Recall that after Proposition 3.1.12 we set β as an element of $\mathbf{E}$ belonging to the $(3B)$ conjugacy class of $\mathbf{M}_\mathbf{c}$. We let $\tilde{\beta}$ be an element of $\widehat{\mathbf{M}_\mathbf{c}}$ whose image in $\mathbf{M}_\mathbf{c}$ is β. By (3.1.13), we have $\mathrm{C}_\mathbf{M}(\beta) = \mathrm{A}_4$, where $\mathrm{A}_4 \cong \mathfrak{A}_4$. It follows from Lemma 3.13.3 that $\mathrm{C}_{\widehat{\mathbf{M}_\mathbf{c}}}(\tilde{\beta})$ is a subgroup of $(\widehat{\mathbf{E}}).\mathrm{A}_4$ of index 3. In particular $\tilde{\beta}$ lies in an $\mathbf{M}$-orbit of length 180.

Since $[\mathbf{A}, \mathbf{M}] = 1$, we have $\mathrm{Orb}(\mathbf{M}, \mathbf{A}^\#) = \{1^2\}$. Thus

$$\mathrm{Orb}(\mathbf{M}, \widehat{\mathbf{E}}^\#) = \{1^2, 20^3, 180\}.$$

As $\mathbf{M}$ has 6 orbits on $\widehat{\mathbf{E}}^\#$, it also has 6 orbits on $\mathrm{Irr}(\widehat{\mathbf{E}})^\#$. By Proposition 3.1.12, the set $\mathrm{Irr}(\widehat{\mathbf{E}} \bmod \mathbf{A})^\#$ accounts for 2 of these orbits. There exists a Galois automorphism γ of the characters of subgroups of $\widehat{\mathbf{E}}$ such that $\rho^\gamma = \rho^2$. Hence γ sends $\mathrm{Irr}(\widehat{\mathbf{E}} \mid \rho)$ bijectively onto the set $\mathrm{Irr}(\widehat{\mathbf{E}} \mid \rho^2)$. In particular these two sets have the same cardinality. Since $\mathrm{Irr}(E)^\#$ is the disjoint union of these two sets and $\mathrm{Irr}(\mathbf{E})^\#$, all of which are $\mathbf{M}$-invariant, we have

$$\begin{aligned} 6 &= \left|\mathrm{Orb}\big(\mathbf{M}, \mathrm{Irr}(\widehat{\mathbf{E}})\big)^{\#}\right| \\ &= \left|\mathrm{Orb}\big(\mathbf{M}, \mathrm{Irr}(\mathbf{E})^{\#}\big)\right| + \left|\mathrm{Orb}\big(\mathbf{M}, \mathrm{Irr}(\widehat{\mathbf{E}} \mid \rho)\big)\right| + \left|\mathrm{Orb}\big(\mathbf{M}, \mathrm{Irr}(\widehat{\mathbf{E}} \mid \rho^2)\big)\right| \\ &= 2 + 2\left|\mathrm{Orb}\big(\mathbf{M}, \mathrm{Irr}(\widehat{\mathbf{E}} \mid \rho)\big)\right|. \end{aligned}$$

We conclude that $\left|\mathrm{Orb}\big(\mathbf{M}, \mathrm{Irr}(\widehat{\mathbf{E}} \mid \rho)\big)\right| = 2$.

Let T_2 be a Sylow 2-subgroup of $\mathbf{M}$. Then from the Atlas T_2 is a maximal subgroup of $\mathbf{M}$. Moreover T_2 is semi-dihedral of order 16.

Now $\mathrm{C}_{\mathbf{E}}(\mathrm{T}_2) = \{1\}$ by (3.1.13). So $\mathrm{C}_{\widehat{\mathbf{E}}}(\mathrm{T}_2) = \mathbf{A}$. Since $\widehat{\mathbf{E}}$ is an abelian 3-group and T_2 is a $3'$-group, by [**As86**, 24.6] we have $\widehat{\mathbf{E}} = \mathbf{A} \times [\widehat{\mathbf{E}}, \mathrm{T}_2]$. Let $\hat{\psi}_1$ be the unique character in $\mathrm{Irr}(\widehat{\mathbf{E}})$ which is trivial on $[\widehat{\mathbf{E}}, \mathrm{T}_2]$ and restricts to ρ on $\mathbf{A}$. So $\hat{\psi}_1 \in \mathrm{C}_{\widehat{\mathbf{E}}^*}(\mathrm{T}_2)$. Moreover $\mathrm{I}_{\mathbf{M}}(\hat{\psi}_1) = \mathrm{T}_2$, since T_2 is maximal in $\mathbf{M}$. So $\hat{\psi}_1$ lies in an $\mathbf{M}$-orbit of length $720/16 = 45$.

Let N_5 be the normalizer of a Sylow 5-subgroup of $\mathbf{M}$. Then N_5 is a maximal subgroup of $\mathbf{M}$, isomorphic to the Frobenius group $5:4$. As with T_2, we can find $\hat{\psi}_2 \in \mathrm{Irr}(\widehat{\mathbf{E}} \mid \rho)$ such that $\mathrm{I}_{\mathbf{M}}(\hat{\psi}_2) = \mathrm{N}_5$. So $\hat{\psi}_2$ lies in an $\mathbf{M}$-orbit of length $720/20 = 36$.

Since $\mathrm{I}_{\mathbf{M}}(\hat{\psi}_i)$ is a $3'$-group, $\hat{\psi}_i$ extends to $\mathrm{I}_{\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M})}(\hat{\psi}_i)$, for $i = 1, 2$. It is routine to show that $\mathrm{Deg}(\mathrm{T}_2) = \{1^4, 2^3\}$ and $\mathrm{Deg}(\mathrm{N}_5) = \{1^4, 4\}$. Hence by Clifford theory

$$\begin{aligned} &\mathrm{Deg}(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \hat{\psi}_1) = \{45^4, 90^3\}, \qquad \mathrm{Def}_3(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \hat{\psi}_1) = \{5^7\}, \\ &\mathrm{Deg}(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \hat{\psi}_2) = \{36^4, 144\}, \qquad \mathrm{Def}_3(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \hat{\psi}_2) = \{5^5\}. \end{aligned} \tag{3.13.6}$$

Also $\mathrm{Irr}(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \rho)$ is the disjoint union of $\mathrm{Irr}(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \hat{\psi}_1)$ and $\mathrm{Irr}(\mathbf{A}.(\mathbf{E} \rtimes \mathbf{M}) \mid \hat{\psi}_2)$. The results of the lemma now follow. $\square$

Finally we have

PROPOSITION 3.13.7. *The group* $\mathrm{N}_{\widehat{\mathbf{M}}_{\mathbf{c}}}(C_2) = \mathbf{A}.\big(\mathbf{E} \rtimes \mathbf{M}\big)$ *has a unique* 3*-block. This block necessarily induces the principal* 3*-block,* B_0^**, of* $\widehat{\mathbf{M}}_{\mathbf{c}}$*. Hence*

$$\text{(3.13.8)} \qquad \begin{aligned} &\mathrm{k}(C_2, \mathsf{B}_0^*, 5 \mid \rho) = 12, \\ &\mathrm{k}(C_2, \mathsf{B}_0^*, d \mid \rho) = 0, \qquad \textit{for all other values of } d. \end{aligned}$$

PROOF. The first statement follows from the fact that $\widehat{\mathbf{E}}$ is self centralizing in $\widehat{\mathbf{M}}_{\mathbf{c}}$. We obtain (3.13.8) from (3.13.6). $\square$

3.14. The Character Degrees of $\mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$

Recall from Corollary 3.12.5 that $\mathrm{Z}(\hat{\mathbf{F}}) = \hat{\mathbf{F}}' = \mathbf{A} \times \mathrm{Z}^1$ is an $\mathbf{L}$-invariant decomposition of the elementary abelian 3-group $\mathrm{Z}(\hat{\mathbf{F}}) \cong \mathbb{Z}_3^2$.

LEMMA 3.14.1. *If* X *is a subgroup of* $\mathrm{Z}(\hat{\mathbf{F}})$ *of order* 3*, then* $\hat{\mathbf{F}}/\mathrm{X}$ *is an extra-special group of type* 3_+^{1+4}.

PROOF. Let $\mathrm{X} \leq \mathrm{Z}(\hat{\mathbf{F}})$ have order 3. Let Z denote the inverse image of $\mathrm{Z}(\hat{\mathbf{F}}/\mathrm{X})$ in $\hat{\mathbf{F}}$. Then $\mathrm{Z}(\hat{\mathbf{F}}) = \hat{\mathbf{F}}' \leq \mathrm{Z} < \hat{\mathbf{F}}$ because $\mathrm{X} < \hat{\mathbf{F}}'$.

Recall that the subgroup $\mathbf{C}$ of $\mathbf{L}$ acts trivially on $\mathrm{Z}(\hat{\mathbf{F}})$. Hence $\mathbf{C}$ normalizes Z, and $\mathrm{Z}/\mathrm{Z}(\hat{\mathbf{F}})$ is a $\mathbf{C}$-invariant subspace of $\hat{\mathbf{F}}/\mathrm{Z}(\hat{\mathbf{F}})$. But by Lemma 3.12.7 the group $\mathbf{C}$ acts irreducibly on $\hat{\mathbf{F}}/\mathrm{Z}(\hat{\mathbf{F}})$. So $\mathrm{Z} = \mathrm{Z}(\hat{\mathbf{F}})$. It follows that $\mathrm{Z}(\hat{\mathbf{F}}/\mathrm{X}) = \mathrm{Z}(\hat{\mathbf{F}})/\mathrm{X} = \hat{\mathbf{F}}'/\mathrm{X}$. But $(\hat{\mathbf{F}}/\mathrm{X})' = \hat{\mathbf{F}}'/\mathrm{X}$, as $\mathrm{X} \leq \hat{\mathbf{F}}'$. So $\mathrm{Z}(\hat{\mathbf{F}}/\mathrm{X}) = (\hat{\mathbf{F}}/\mathrm{X})'$.

Finally, $\hat{\mathbf{F}}/\mathrm{X}$ has exponent 3, since by the Atlas $\widehat{\mathbf{M}}_{\mathbf{c}}$ has no elements of order 9. $\square$

We can now prove the following

LEMMA 3.14.2. *There exist characters* $\hat{\chi}_1, \hat{\chi}_2 \in \mathrm{Irr}(\hat{\mathbf{F}} \mid \rho)$ *such that* $\mathrm{I}_{\mathbf{A}.(\mathbf{F}\rtimes\mathbf{L})}(\hat{\chi}_1) = \mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$, *and* $\mathrm{I}_{\mathbf{A}.(\mathbf{F}\rtimes\mathbf{L})}(\hat{\chi}_2) = \mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{C}\big)$. *Each* $\hat{\chi}_i$ *extends to its stabilizer in* $\mathbf{A}.(\mathbf{F} \rtimes \mathbf{L})$. *Hence*

(3.14.3)

$$\mathrm{Deg}(\mathbf{A}.(\mathbf{F} \rtimes \mathbf{L}) \mid \hat{\chi}_1) = \{9^2, 36^5, 45^2, 54^3\}, \qquad \mathrm{Def}_3(\mathbf{A}.(\mathbf{F} \rtimes \mathbf{L}) \mid \hat{\chi}_1) = \{5^9, 4^3\},$$

$$\mathrm{Deg}(\mathbf{A}.(\mathbf{F} \rtimes \mathbf{L}) \mid \hat{\chi}_2) = \{18, 36^2, 54^2, 72^2, 90, 108\}, \quad \mathrm{Def}_3(\mathbf{A}.(\mathbf{F} \rtimes \mathbf{L}) \mid \hat{\chi}_2) = \{5^6, 4^3\}.$$

Also

$\mathrm{Irr}(\mathbf{A}.(\mathbf{F} \rtimes \mathbf{L}) \mid \rho)$ *is the disjoint union of* $\mathrm{Irr}(\mathbf{A}.(\mathbf{F} \rtimes \mathbf{L}) \mid \hat{\chi}_1)$ *and* $\mathrm{Irr}(\mathbf{A}.(\mathbf{F} \rtimes \mathbf{L}) \mid \hat{\chi}_2)$.

PROOF. Let $\lambda \in \mathrm{Irr}(\mathrm{Z}(\hat{\mathbf{F}}))^{\#}$. We can regard λ as an element of $\mathrm{Irr}(\mathrm{Z}(\hat{\mathbf{F}})/\operatorname{Ker}(\lambda))^{\#}$. Then by Theorem 1.2.18, there is unique irreducible character $\hat{\lambda}$ of $\hat{\mathbf{F}}/\operatorname{Ker}(\lambda)$ lying over λ, and $\hat{\lambda}$ has degree 9 and vanishes outside $\mathrm{Z}(\hat{\mathbf{F}})/\operatorname{Ker}(\lambda)$. By inflation, we may regard $\hat{\lambda}$ as an irreducible character of $\hat{\mathbf{F}}$. We see in this way that $\hat{\mathbf{F}}$ has eight characters of degree 9 which vanish outside $\mathrm{Z}(\hat{\mathbf{F}})$. Since $\mathrm{Irr}(\hat{\mathbf{F}} \mid \rho)$ and $\mathrm{Irr}(\hat{\mathbf{F}} \mid \rho^2)$ are of the same size, and account for all irreducible characters of $\mathrm{Irr}(\hat{\mathbf{F}} \mid \mathrm{Z}(\hat{\mathbf{F}}))$ non-trivial on $\mathbf{A}$, both of these sets contain $(8-2)/2 = 3$ characters.

We pick $\hat{\chi}_1, \hat{\chi}_2 \in \mathrm{Irr}(\hat{\mathbf{F}} \mid \rho)$, with $\hat{\chi}_1$ trivial on Z^1 and $\hat{\chi}_2$ non-trivial on Z^1. Then $\mathrm{I}_{\mathbf{A}.(\mathbf{F}\rtimes\mathbf{L})}(\hat{\chi}_1) = \mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$, and $\mathrm{I}_{\mathbf{A}.(\mathbf{F}\rtimes\mathbf{L})}(\hat{\chi}_2) = \mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{C}\big)$. Since $\mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)/\hat{\mathbf{F}} \cong \mathbf{L}$ has cyclic Sylow 3-subgroups, both $\hat{\chi}_1$ and $\hat{\chi}_2$ extend to their stabilizers in $\mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$. We obtain (3.14.3) immediately using Clifford theory.

The $\mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$-invariant character $\hat{\chi}_1$ and the two $\mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$-conjugates of $\hat{\chi}_2$ account for all three elements of $\mathrm{Irr}(\hat{\mathbf{F}} \mid \rho)$. □

To conclude our analysis of $\mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$ we now prove

PROPOSITION 3.14.4. *The group* $\mathrm{N}_{\widehat{\mathbf{M}}_{\mathbf{c}}}(C_4) = \mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$ *has a unique* 3*-block, which necessarily induces the principal* 3*-block,* B_0^**, of* $\widehat{\mathbf{M}}_{\mathbf{c}}$*. Hence*

$$\begin{aligned} &\mathrm{k}(C_4, \mathsf{B}_0^*, 5 \mid \rho) = 15, \\ (3.14.5) \qquad &\mathrm{k}(C_4, \mathsf{B}_0^*, 4 \mid \rho) = 6, \\ &\mathrm{k}(C_4, \mathsf{B}_0^*, d \mid \rho) = 0, \qquad \textit{for all other values of } d. \end{aligned}$$

PROOF. Since $\mathrm{C}_{\mathbf{A}.(\mathbf{F}\rtimes\mathbf{L})}(\hat{\mathbf{F}}) = \mathrm{Z}(\hat{\mathbf{F}})$, the group $\mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$ has a unique 3-block. Then (3.14.5) follows from (3.14.3). □

3.15. The Character Degrees of $\mathbf{A}.\big(\mathbf{W} \rtimes \mathbf{Q}\big)$

Most of the work of this section has already been done in Section 3.14. We have

LEMMA 3.15.1. *There exist* $\hat{\chi}_1, \hat{\chi}_2 \in \mathrm{Irr}(\hat{\mathbf{F}})$ *such that* $\mathrm{I}_{\mathbf{A}.(\mathbf{W}\rtimes\mathbf{Q})}(\hat{\chi}_1) = \mathbf{A}.\big(\mathbf{W} \rtimes \mathbf{Q}\big)$ *and* $\mathrm{I}_{\mathbf{A}.(\mathbf{W}\rtimes\mathbf{Q})}(\hat{\chi}_2) = \mathbf{A}.\big(\mathbf{W} \rtimes \langle b \rangle\big)$*, where* b *is an element of* $\mathbf{Q} \cap \mathbf{C}$ *of order* 4*. Hence*

$$\begin{aligned} &\mathrm{Deg}(\mathbf{A}.(\mathbf{W} \rtimes \mathbf{Q}) \mid \hat{\chi}_1) = \{9^4, 18^5\}, \qquad \mathrm{Def}_3(\mathbf{A}.(\mathbf{W} \rtimes \mathbf{Q}) \mid \hat{\chi}_1) = \{5^9\}, \\ (3.15.2) \qquad &\mathrm{Deg}(\mathbf{A}.(\mathbf{W} \rtimes \mathbf{Q}) \mid \hat{\chi}_2) = \{18^4, 36^2\}, \qquad \mathrm{Def}_3(\mathbf{A}.(\mathbf{W} \rtimes \mathbf{Q}) \mid \hat{\chi}_2) = \{5^6\}. \end{aligned}$$

Also

$\mathrm{Irr}(\mathbf{A}.(\mathbf{W}\rtimes\mathbf{Q}) \mid \rho)$ *is the disjoint union of* $\mathrm{Irr}(\mathbf{A}.(\mathbf{W}\rtimes\mathbf{Q}) \mid \hat{\chi}_1)$ *and* $\mathrm{Irr}(\mathbf{A}.(\mathbf{W}\rtimes\mathbf{Q}) \mid \hat{\chi}_2)$.

PROOF. The existence of $\hat{\chi}_1$ and $\hat{\chi}_2$ comes from Lemma 3.14.2. The character $\hat{\chi}_1$ is invariant in $\mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{L}\big)$, and hence in $\mathbf{A}.\big(\mathbf{W} \rtimes \mathbf{Q}\big)$. The stabilizer of $\hat{\chi}_2$ in $\mathbf{A}.\big(\mathbf{W} \rtimes \mathbf{Q}\big)$ is $\mathbf{A}.\big(\mathbf{W} \rtimes \mathbf{Q}\big) \cap \mathbf{A}.\big(\mathbf{F} \rtimes \mathbf{C}\big) = \mathbf{A}.\big(\mathbf{F} \rtimes (\mathbf{S}_{\mathbf{L}} \rtimes \langle b \rangle)\big)$ It also follows immediately from

Lemma 3.14.2 that both $\hat{\chi}_1$ and $\hat{\chi}_2$ extend to their stabilizers in $\mathbf{A}.(\mathbf{W} \rtimes \mathbf{Q})$. We obtain (3.15.2) using (3.4.1), (3.4.4) and Clifford Theory. $\square$

We can now prove the following

PROPOSITION 3.15.3. *The group* $\mathrm{N}_{\widehat{\mathbf{M}}_{\mathbf{c}}}(C_3) = \mathbf{A}.(\mathbf{W} \rtimes \mathbf{Q})$ *has a unique* 3*-block. This block necessarily induces the principal* 3*-block,* B_0^**, of* $\widehat{\mathbf{M}}_{\mathbf{c}}$*. Hence*

$$\text{(3.15.4)} \qquad \begin{aligned} &\mathrm{k}(C_3, \mathsf{B}_0^*, 5 \mid \rho) = 15, \\ &\mathrm{k}(C_3, \mathsf{B}_0^*, d \mid \rho) = 0, \qquad \textit{for all other values of } d. \end{aligned}$$

PROOF. The group $\mathbf{A}.(\mathbf{W} \rtimes \mathbf{Q})$ has a unique 3-block since $\mathrm{C}_{\mathbf{A}.(\mathbf{W} \rtimes \mathbf{Q})}(\mathbf{W}) = \mathrm{Z}(\widehat{\mathbf{W}})$. Then (3.15.4) follows from (3.15.2). $\square$

3.16. The Projective Conjecture for the prime $p = 3$

The universal covering group $\widehat{\mathbf{M}}_{\mathbf{c}}$ of $\mathbf{M}_{\mathbf{c}}$ has a single 3-block of defect 7, namely the principal block B_0^*. All other 3-blocks are of defect 1. We list here the characters of B_0^* lying over some fixed $\rho \in \mathrm{Irr}(\mathbf{A})^{\#}$ and their 3-defects:

Character	χ_{25}	χ_{26}	χ_{27}	χ_{28}	χ_{29}	χ_{30}	χ_{31}	χ_{32}	χ_{33}
Degree	126	126	792	1980	2376	2376	2520	2520	2772
3-Defect	5	5	5	5	4	4	5	5	5
Character	χ_{34}	χ_{36}	χ_{37}	χ_{38}	χ_{41}	χ_{42}	χ_{43}	χ_{44}	χ_{45}
Degree	4752	6336	6336	7875	8064	10395	10395	10395	12375
3-Defect	4	5	5	5	5	4	4	4	5

Thus

$$
\begin{aligned}
&\mathrm{k}(C_1, \mathsf{B}_0^*, 5 \mid \rho) = 12, \\
\text{(3.16.1)} \qquad &\mathrm{k}(C_1, \mathsf{B}_0^*, 4 \mid \rho) = 6, \\
&\mathrm{k}(C_1, \mathsf{B}_0^*, d \mid \rho) = 0, \qquad \text{for all other values of } d.
\end{aligned}
$$

THEOREM 3.16.2. *The Projective Conjecture holds for the McLaughlin Simple Group and the prime $p = 3$.*

PROOF. From Conjecture 1.4.6 and Table 3.1 on page 48 we need to prove

$$\mathrm{k}(C_1, \mathsf{B}_0^*, d \mid \rho) + \mathrm{k}(C_3, \mathsf{B}_0^*, d \mid \rho) = \mathrm{k}(C_2, \mathsf{B}_0^*, d \mid \rho) + \mathrm{k}(C_4, \mathsf{B}_0^*, d \mid \rho),$$

for all values of $d \in \mathbb{Z}_+$. From (3.13.8), (3.14.5), (3.15.4) and (3.16.1) we obtain the following sums for the equation above for various values of d:

3-Defect	C_1		C_3		C_2		C_4
5	12	+	15	=	12	+	15
4	6	+	0	=	0	+	6

TABLE 3.4. The Projective Conjecture for $p = 3$

The summands in the above equation are zero for all other values of d. This proves the Projective Conjecture for McLaughlin's simple group for the prime $p = 3$. □

Chapter 4

The Prime p=2

4.1. The 2-local structure of $\mathbf{M_c}$

We begin with a fundamental property of $\mathbf{M_c}$.

PROPOSITION 4.1.1. $\mathbf{M_c}$ *has exactly one class of involutions and the centralizer of an involution is isomorphic to* $2.\mathfrak{A}_8$.

PROOF. In [**JW72**] Janko and Wong characterised the McLaughlin Group as the unique finite simple group possessing a single class of involutions, with the centralizer of an involution isomorphic to the covering group $2.\mathfrak{A}_8$ of $\mathfrak{A}_8$. □

For the rest of the thesis τ will denote a fixed involution of $\mathbf{M_c}$ and $\mathbf{H}$ will denote the centralizer $\mathrm{C}_{\mathbf{M_c}}(\tau)$ of τ in $\mathbf{M_c}$. The group $\langle\tau\rangle$ will be denoted by $\mathbf{T}$.

LEMMA 4.1.2. *The group* $\mathbf{H}/\mathbf{T} \cong \mathfrak{A}_8$ *has exactly two conjugacy classes of involutions: the class* $(2A)$ *consisting of all involutions of cycle structure* $(..)(..)(..)(..)$, *and the class* $(2B)$ *consisting of all involutions of cycle structure* $(..)(..)$. *The inverse image in* $\mathbf{H}$ *of the class* $(2A)$ *is the unique conjugacy class of non-central involutions of* $\mathbf{H}$, *while that of* $(2B)$ *is a single conjugacy class of elements of order* 4 *in* $\mathbf{H}$.

PROOF. This comes from the character table of $\mathfrak{A}_8$ found in the Atlas. □

We now prove

PROPOSITION 4.1.3. $\mathbf{M_c}$ *has exactly one conjugacy class of elementary abelian* 2-*subgroups of order* 2^2. *If* K *is a representative of one of these classes then* $\mathrm{N}_{\mathbf{M_c}}(\mathrm{K})$ *acts doubly transitively on* $\mathrm{K}^{\#}$.

PROOF. It is clear from Lemma 4.1.2 that $\mathbf{H}$, and hence $\mathbf{M_c}$, possesses elementary abelian 2-groups of order 2^2.

Suppose $\mathrm{K}_1, \mathrm{K}_2$ are elementary abelian 2-subgroups of $\mathbf{M_c}$ of order 2^2. Choose arbitrary pairs of distinct elements $(\tau_1, \tilde{\varepsilon}_1)$ and $(\tau_2, \tilde{\varepsilon}_2)$ from $\mathrm{K}_1^{\#}$ and $\mathrm{K}_2^{\#}$ respectively. Note that $\mathrm{K}_1 = \langle \tau_1, \tilde{\varepsilon}_1 \rangle$ and $\mathrm{K}_2 = \langle \tau_2, \tilde{\varepsilon}_2 \rangle$.

In view of Proposition 4.1.1, the elements τ_1, τ_2 and τ are all conjugate in $\mathbf{M_c}$. Since we are interested in K_1 and K_2 only up to conjugacy in $\mathbf{M_c}$, we may assume, and we do, that $\tau_1 = \tau_2 = \tau$. Hence both K_1 and K_2 are subgroups of $\mathbf{H} = \mathrm{C}_{\mathbf{M_c}}(\tau)$.

By Lemma 4.1.2, there exists $x \in \mathbf{H}$ such that $\tilde{\varepsilon}_1^x = \tilde{\varepsilon}_2$. Obviously $\tau^x = \tau$. So $\mathrm{K}_1^x = \mathrm{K}_2$. This shows that $\mathbf{M_c}$ has a single conjugacy class of elementary abelian 2-subgroups of order 2^2. Taking $\mathrm{K}_1 = \mathrm{K}_2$ to begin with, the above argument also shows that $\mathrm{N}_{\mathbf{M_c}}(\mathrm{K}_1)$ acts doubly transitively on $\mathrm{K}_1^{\#}$. $\square$

We collect information on certain classes of 2-subgroups of $\mathfrak{A}_8$, which we will use to classify the maximal elementary abelian 2-subgroups of $\mathbf{H}$.

LEMMA 4.1.4. *Let* A_0 *be a regular transitive embedding of* $\mathbb{Z}_2^3$ *in* $\mathfrak{A}_8$. *Then* A_0 *is a self-centralizing trivial intersection subgroup of* $\mathfrak{A}_8$, *and all elements of* $\mathrm{A}_0^{\#}$ *come from the* (2*A*) *conjugacy class of* $\mathfrak{A}_8$. *The normalizer* $\mathrm{N}_{\mathfrak{A}_8}(\mathrm{A}_0)$ *of* A_0 *in* $\mathfrak{A}_8$ *is isomorphic* *to the holomorph* $\mathrm{A}_0 \rtimes \mathrm{Aut}(\mathrm{A}_0)$ *of* A_0*. There are exactly two* $\mathfrak{A}_8$*-conjugacy classes of such subgroups* A_0*. These two classes fuse in* $\mathfrak{S}_8$*.*

Proof. Since A_0 is an abelian regular transitive subgroup of $\mathfrak{S}_8$, it is self-centralizing in $\mathfrak{S}_8$, and all its non-trivial elements come from the $(2A)$ conjugacy class of $\mathfrak{A}_8$ consisting of involutions of cycle structure $(..)(..)(..)(..)$. Furthermore $\mathrm{N}_{\mathfrak{S}_8}(\mathrm{A}_0)$ is the holomorph of A_0 i.e., the semi-direct product of A_0 with its automorphism group $\mathrm{Aut}(\mathrm{A}_0) \cong \mathrm{GL}(3,2)$. It is clear that $\mathrm{N}_{\mathfrak{S}_8}(\mathrm{A}_0)$ is contained in $\mathfrak{A}_8$ (for instance because the simple group $\mathrm{GL}(3,2)$ has no subgroups of index 2). Hence there are two $\mathfrak{A}_8$-conjugacy classes of subgroups of this type. However the full symmetric group $\mathfrak{S}_8$ contains only one such class.

From the Atlas, the conjugacy class $(2A)$ of $\mathfrak{A}_8$ contains 105 elements. Every element in this class is contained in some $\mathfrak{A}_8$-conjugate of A_0. From the structure of $\mathrm{N}_{\mathfrak{A}_8}(\mathrm{A}_0)$, the group A_0 has $(8!/2)/(2^3 \cdot 168) = 15$ conjugates in $\mathfrak{A}_8$. So the $\mathfrak{A}_8$-conjugates of A_0 contain at most $|\mathrm{A}_0^{\#}| \cdot |\mathfrak{A}_8 : \mathrm{A}_0| = 7 \cdot 15 = 105$ non-trivial elements. Hence every element of the class $(2A)$ is contained in exactly one conjugate of A_0. It follows that distinct $\mathfrak{A}_8$-conjugates of A_0 intersect trivially. □

We now discuss the covering group of A_0 in $\mathbf{H}$.

Lemma 4.1.5. *Let* A_0 *be a regular transitive embedding of* $\mathbb{Z}_2^3$ *in* $\mathfrak{A}_8$*. Then the covering group* $\mathbf{R}_0$ *of* A_0 *in* $\mathbf{H}$ *is elementary abelian of order* 2^4*, and is a self-centralizing subgroup of* $\mathbf{M_c}$*. Distinct* $\mathbf{H}$*-conjugates of* $\mathbf{R}_0$ *intersect in* $\mathbf{T}$*. Also* $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_0) = \mathbf{R}_0 \rtimes \mathrm{G}_0$*, where* $\mathrm{G}_0 \cong \mathrm{GL}(3,2)$ *acts as the full automorphism group on* $\mathrm{A}_0 = \mathbf{R}_0/\mathbf{T}$*.*

Proof. By Lemmas 4.1.2 and 4.1.4, all elements of $\mathbf{R}_0 \backslash \mathbf{T}$ come from the unique conjugacy class of non-central involutions of $\mathbf{H}$. In particular $\mathbf{R}_0$ has exponent 2. So $\mathbf{R}_0$ is an elementary abelian 2-group of order 2^4.

Since A_0 is a self-centralizing subgroup of $\mathfrak{A}_8$, it follows that the abelian group $\mathbf{R}_0$ is self-centralizing in $\mathbf{H}$. But $\mathrm{C}_{\mathbf{M_c}}(\mathbf{R}_0) \le \mathbf{H} = \mathrm{C}_{\mathbf{M_c}}(\tau)$. So $\mathbf{R}_0$ is also self-centralizing in $\mathbf{M_c}$.

Distinct $\mathbf{H}$-conjugates of $\mathbf{R}_0$ intersect in $\mathbf{T}$, since A_0 is a trivial intersection subgroup of $\mathfrak{A}_8 \cong \mathbf{H}/\mathbf{T}$, and each $\mathbf{H}$-conjugate contains $\mathbf{T}$.

As $\mathrm{N}_{\mathfrak{A}_8}(\mathrm{A}_0) \cong \mathrm{A}_0 \rtimes \mathrm{GL}(3,2)$, it follows that $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_0) \cong \mathbf{R}_0 .\, \mathrm{GL}(3,2)$, where the factor $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_0)/\mathbf{R}_0 \cong \mathrm{GL}(3,2)$ acts as the full automorphism group on $\mathrm{A}_0 = \mathbf{R}_0/\mathbf{T}$.

To show that $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_0)$ splits over $\mathbf{R}_0$, we need only show that one of its Sylow 2-subgroups splits over $\mathbf{R}_0$. To this end we fix

$$\mathrm{A}_0 = \big\langle (1,2)(3,4)(5,6)(7,8), \quad (1,3)(2,4)(5,7)(6,8), \quad (1,5)(2,6)(3,7)(4,8) \big\rangle,$$

$$x = (1,3)(2,4)(5,8)(6,7), \quad y = (1,5)(2,6)(3,8)(4,7), \quad z = (1,2)(3,4)(5,7)(6,8),$$

and set $\mathrm{K} = \langle x, y \rangle$ and $\mathrm{D}_8 = \mathrm{K} \rtimes \langle z \rangle$. Then $\mathrm{K} \cong \mathbb{Z}_2^2$ while $\mathrm{D}_8 \cong D_8$, and the group $\mathrm{S}_2 = \mathrm{A}_0 \rtimes \mathrm{D}_8$ is a Sylow 2-subgroup of $\mathrm{N}_{\mathfrak{A}_8}(\mathrm{A}_0)$.

Let $\tilde{x}, \tilde{y}$ and $\tilde{z}$ be elements of $\mathbf{H}$ whose images in $\mathbf{H}/\mathbf{T}$ are x, y and z respectively. Then $\tilde{x}, \tilde{y}$ and $\tilde{z}$ are involutions and $\tilde{x}$ commutes with both $\tilde{y}$ and $\tilde{z}$. If $\tilde{y}^{\tilde{z}} = \tilde{x}\tilde{y}$ then $\langle \tilde{x}, \tilde{y} \rangle \rtimes \langle \tilde{z} \rangle \cong D_8$ is a complement to $\mathbf{T}$ in $\mathbf{T}.\mathrm{D}_8$. Otherwise $\tilde{y}^{\tilde{z}} = \tau\tilde{x}\tilde{y}$ and $\langle \tau\tilde{x}, \tilde{y} \rangle \rtimes \langle \tilde{z} \rangle \cong D_8$ is a complement to $\mathbf{T}$ in $\mathbf{T}.\mathrm{D}_8$. In any event $\mathbf{T}.\mathrm{D}_8$ splits over $\mathbf{T}$. But this shows that the Sylow 2-subgroup $\mathbf{T}.\mathrm{S}_2 = \mathbf{R}_0.\mathrm{D}_8$ of $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_0)$ splits over $\mathbf{R}_0$. The rest of the lemma now follows. $\square$

COROLLARY 4.1.6. $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_0)$ *acts transitively on* $\mathbf{R}_0\backslash\mathbf{T}$.

PROOF. Given $x, y \in \mathbf{R}_0\backslash\mathbf{T}$, there exists $z \in \mathbf{H}$ such that $x^z = y$, by Lemma 4.1.2. Then $\mathbf{R}\cap\mathbf{R}^z \geq \langle\tau, y\rangle > \mathbf{T}$. It follows from Lemma 4.1.5 that $\mathbf{R}_0 = \mathbf{R}_0^z$. Hence in fact $z \in \mathrm{N}_{\mathbf{H}}(\mathbf{R}_0)$. This proves the corollary. □

The next lemma shows why we are interested in the two conjugacy classes of subgroups of $\mathbf{H}$ of type $\mathbf{R}_0$.

LEMMA 4.1.7. *Every elementary abelian* 2*-subgroup of* $\mathbf{H}$ *is contained in the inverse image* $\mathbf{R}_0 \cong \mathbb{Z}_2^4$ *of some regular transitive embedding* A_0 *of* $\mathbb{Z}_2^3$ *in* $\mathbf{H}/\mathbf{T} \cong \mathfrak{A}_8$. *Hence* $\mathbf{H}$ *has exactly two conjugacy classes of maximal elementary abelian* 2*-subgroups. These two classes fuse in* $\mathbf{H}.2 = \mathrm{C}_{\mathbf{M}_c.2}(\tau)$.

PROOF. Let X be an elementary abelian 2-subgroup of $\mathbf{H}$, and let $\overline{\mathrm{X}}$ denote the group $\mathrm{X}\mathbf{T}/\mathbf{T}$. Recall from Lemma 4.1.2 that the single conjugacy class of non-central involutions of $\mathbf{H}$ lies over the conjugacy class $(2A)$ consisting of involutions of $\mathbf{H}/\mathbf{T} \cong \mathfrak{A}_8$ of cycle structure (..)(..)(..)(..). Hence no non-trivial element of $\overline{\mathrm{X}}$ fixes a point. It follows that $\overline{\mathrm{X}}$ acts regularly on each of its orbits. In particular each orbit of $\overline{\mathrm{X}}$ has length $|\overline{\mathrm{X}}|$.

There is a single conjugacy class of elementary abelian 2-subgroups of $\mathfrak{S}_8$ of order $|\overline{\mathrm{X}}|$ which act regularly on their orbits. Moreover a regular transitive embedding of $\mathbb{Z}_2^3$ in $\mathfrak{S}_8$ contains representatives of each such class. We conclude that $\overline{\mathrm{X}} \leq \mathrm{A}_0$, where A_0 is a member of one of the two classes of regular transitive embeddings of $\mathbb{Z}_2^3$ in $\mathfrak{A}_8$ given by Lemma 4.1.4.

The previous paragraph shows that $X \leq \mathbf{R}_0$, where $\mathbf{R}_0$ is the inverse image of A_0 in $\mathbf{H}$. But by Lemma 4.1.5, the group $\mathbf{R}_0$ is itself elementary abelian of order 2^4. Hence $\mathbf{H}$ has two conjugacy classes of maximal elementary abelian 2-subgroups, and a member of either class is the inverse image of a regular transitive embedding of $\mathbb{Z}_2^3$ in $\mathfrak{A}_8$.

From the Atlas $C_{\mathbf{M_c}.2}(\tau) = \mathbf{H}.2 \cong 2.\mathfrak{S}_8$. It then follows from Lemmas 4.1.4 and 4.1.5 that the two classes of maximal elementary abelian 2-subgroups of $\mathbf{H}$ fuse in $\mathbf{H}.2$. $\square$

For the rest of this section we let $\mathbf{R}_1$ and $\mathbf{R}_2$ be fixed representatives of the two classes of maximal elementary abelian 2-subgroups of $\mathbf{H}$ of order 2^4. If we prove any result for $\mathbf{R}_1$, then an analogous result will hold for $\mathbf{R}_2$. In view of Lemma 4.1.7, the factor group $A_1 = \mathbf{R}_1/\mathbf{T}$ is a regular transitive embedding of $\mathbb{Z}_2^3$ in $\mathbf{H}/\mathbf{T} \cong \mathfrak{A}_8$.

Lemma 4.1.8. $N_{\mathbf{M_c}}(\mathbf{R}_1) = \mathbf{R}_1 \rtimes \Lambda_1$, *where* $\Lambda_1 \cong \mathfrak{A}_7$ *acts flag transitively on* $\mathbf{R}_1$.

Proof. Let $\tilde{\varepsilon}$ be any element of $\mathbf{R}_1 \backslash \mathbf{T}$. By Proposition 4.1.1 there exists $x \in \mathbf{M_c}$ such that $\tau^x = \tilde{\varepsilon}$. Hence $\mathbf{R}_1 \leq C_{\mathbf{M_c}}(\tilde{\varepsilon}) = \mathbf{H}^x$ is in one of the two classes of maximal elementary abelian 2-subgroups of $\mathbf{H}^x$ given by Lemma 4.1.7. Then by Corollary 4.1.6, all elements of $\mathbf{R}_1 \backslash \langle \tilde{\varepsilon} \rangle$ are conjugate in $N_{\mathbf{H}^x}(\mathbf{R}_1)$, and all elements of $\mathbf{R}_1 \backslash \mathbf{T}$ are conjugate in $N_{\mathbf{H}}(\mathbf{R}_1)$. It follows that $N_{\mathbf{M_c}}(\mathbf{R}_1)$ acts transitively on $\mathbf{R}_1^{\#}$. In parlicular $N_{\mathbf{H}}(\mathbf{R}_1)$ is a subgroup of index $|\mathbf{R}_1^{\#}| = 15$ in $N_{\mathbf{M_c}}(\mathbf{R}_1)$.

From Lemma 4.1.5 we have $N_{\mathbf{H}}(\mathbf{R}_1) = \mathbf{R}_1 \rtimes G_1$, where $G_1 \cong GL(3,2)$ acts as the full automorphism group on $\mathbf{R}_1/\mathbf{T}$. But $N_{\mathbf{H}}(\mathbf{R}_1)$ is the centralizer of the element

$\tau \in \mathbf{R}_1^{\#}$ in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$. We deduce from this, and the transitivity of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$ on $\mathbf{R}_1^{\#}$, that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$ acts flag transitively on $\mathbf{R}_1$.

Now $\mathbf{R}_1$ is a self-centralizing subgroup of $\mathbf{M_c}$, by Lemma 4.1.5. Hence $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)/\mathbf{R}_1$ is isomorphic to a subgroup of index $|\mathfrak{A}_8|/\big(|\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1) : \mathrm{N}_{\mathbf{H}}(\mathbf{R}_1)| \cdot |\mathrm{N}_{\mathbf{H}}(\mathbf{R}_1)|\big) = (8!/2)/(15 \cdot 168) = 8$ in $\mathrm{Aut}(\mathbf{R}_1) \cong \mathrm{GL}(4,2) \cong \mathfrak{A}_8$. From the list of maximal subgroups of $\mathfrak{A}_8$ found in the Atlas, it follows that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)/\mathbf{R}_1 \cong \mathfrak{A}_7$.

By Lemma 4.1.5, the group $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_1)$ splits over $\mathbf{R}_1$. It also contains a Sylow 2-subgroup of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$. Hence the latter group also splits over $\mathbf{R}_1$. This completes the proof of the lemma. $\square$

COROLLARY 4.1.9. $\mathbf{M_c}$ *has exactly two conjugacy classes of maximal elementary abelian* 2*-groups, corresponding to the two classes of subgroups of* $\mathbf{H}$ *isomorphic to* $\mathbb{Z}_2^4$*. The former two classes fuse in* $\mathbf{M_c}.2$*.*

PROOF. It suffices to show that $\mathbf{R}_1$ is conjugate to $\mathbf{R}_2$ in $\mathbf{M_c}.2$ but not in $\mathbf{M_c}$, since any maximal elementary abelian 2-subgroup of $\mathbf{M_c}$ is $\mathbf{M_c}$-conjugate to a maximal elementary abelian 2-subgroup of $\mathbf{H}$, and hence to either $\mathbf{R}_1$ or $\mathbf{R}_2$.

That $\mathbf{R}_1$ and $\mathbf{R}_2$ are conjugate in $\mathbf{M_c}.2$ is a consequence of Lemma 4.1.7.

Suppose $\mathbf{R}_1 = \mathbf{R}_2^x$, for some $x \in \mathbf{M_c}$. Then $\tau^x \in \mathbf{R}_1$, as $\tau \in \mathbf{R}_2$. Since both τ and τ^x lie in $\mathbf{R}_1$, it follows from Lemma 4.1.8 that there exists $y \in \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$ such that $\tau^{xy} = \tau$. Hence $\mathbf{R}_1 = \mathbf{R}_2^{xy}$, where $xy \in \mathbf{H} = \mathrm{C}_{\mathbf{M_c}}(\tau)$. This contradicts our choice of $\mathbf{R}_1$ and $\mathbf{R}_2$. We conclude that $\mathbf{R}_1$ and $\mathbf{R}_2$ are not conjugate in $\mathbf{M_c}$. $\square$

COROLLARY 4.1.10. *Let* S *be a Sylow* 2*-subgroup of* $\mathbf{M_c}$*. Then* S *contains exactly one* $\mathbf{M_c}$*-conjugate of* $\mathbf{R}_1$*. In particular* $\mathbf{R}_1$ *is weakly closed in* S *with respect to* $\mathbf{M_c}$*.*

PROOF. We may assume that S is a Sylow 2-subgroup of $\mathbf{H}$ containing $\mathbf{R}_1$. Let $\mathbf{R}_1^{(1)}$ be an $\mathbf{M_c}$-conjugate of $\mathbf{R}_1$ contained in S, and set $\mathrm{A}_1^{(1)} = \mathbf{R}_1^{(1)}/\mathbf{T}$. We need to show that $\mathbf{R}_1 = \mathbf{R}_1^{(1)}$.

The two conjugacy classes of elementary abelian 2-subgroups of $\mathbf{H}$ of order 2^4 do not fuse in $\mathbf{M_c}$, by Corollary 4.1.9. So $\mathbf{R}_1$ and $\mathbf{R}_1^{(1)}$ are conjugate in $\mathbf{H}$. Also $\mathrm{Z}(\mathrm{S}/\mathbf{T})$ is contained in $\mathrm{A}_1 \cap \mathrm{A}_1^{(1)}$, since by Lemma 4.1.4 both A_1 and $\mathrm{A}_1^{(1)}$ are self-centralizing in $\mathbf{H}/\mathbf{T}$. But A_1 is a trivial intersection subgroup of $\mathbf{H}/\mathbf{T}$, by the same lemma. So $\mathrm{A}_1 = \mathrm{A}_1^{(1)}$. We conclude that $\mathbf{R}_1 = \mathbf{R}_1^{(1)}$. $\square$

The next result is due to Larry Finkelstein [**Fk73**, Lemma 5.3].

COROLLARY 4.1.11. *If* X *is a subgroup of* $\mathbf{R}_1$ *properly containing* $\mathbf{T}$ *then* $\mathrm{N}_{\mathbf{M_c}}(\mathrm{X}) \leq \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$.

PROOF. Suppose $x \in \mathrm{N}_{\mathbf{M_c}}(\mathrm{X})$. Then $\mathbf{T}^x \leq \mathrm{X}^x = \mathrm{X} \leq \mathbf{R}_1$. By Lemma 4.1.8 we can find $y \in \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$ such that $\mathbf{T}^{xy} = \mathbf{T}$ and $\mathrm{X}^{xy} = \mathrm{X}$. So $xy \in \mathrm{N}_{\mathbf{H}}(\mathrm{X})$ and $\mathbf{T} < \mathrm{X} \leq \mathbf{R}_1 \cap \mathbf{R}_1^{xy}$. We conclude from Lemma 4.1.5 that xy normalizes $\mathbf{R}_1$. Hence so too does x. This proves the corollary. $\square$

We now prove the most important result of this section.

PROPOSITION 4.1.12. *If* $\mathbf{R}$ *is a non-trivial radical* 2*-subgroup of* $\mathbf{M_c}$*, then* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R})$ *is conjugate in* $\mathbf{M_c}$ *to a subgroup of* $\mathbf{H}$*,* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$ *or* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_2)$.

PROOF. Since $\mathbf{R}$ is non-trivial, there is some involution τ^1 in $\mathrm{Z}(\mathbf{R})$. After a $\mathbf{M_c}$-conjugation, we may assume that $\tau^1 = \tau$. Then $\mathbf{T} = \langle \tau \rangle \leq \mathrm{Z}(\mathbf{R}) \leq \mathbf{R}$ implies that

$\mathbf{R} \leq \mathbf{H} = \mathrm{C}_{\mathbf{M_c}}(\tau)$. Let $\mathrm{X} = \Omega\mathrm{Z}(\mathbf{R})$. Then $\mathbf{T} \leq \mathrm{X}$, and by Corollary 4.1.9 the group X is contained in a $\mathbf{H}$-conjugate of either $\mathbf{R}_1$ or $\mathbf{R}_2$.

If $\mathbf{T} = \mathrm{X}$, then $\mathrm{N}_{\mathbf{M_c}}(\mathrm{X}) = \mathbf{H}$. If $\mathbf{T} < \mathrm{X}$, then $\mathrm{N}_{\mathbf{M_c}}(\mathrm{X})$ is contained in a $\mathbf{H}$-conjugate of either $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_1)$ or $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_2)$, by Corollary 4.1.11. The proposition now follows from the fact that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}) \leq \mathrm{N}_{\mathbf{M_c}}(\mathrm{X})$. $\square$

The easiest way to describe the radical 2-subgroups of $\mathbf{H}/\mathbf{T}$ is to use the $\mathrm{GL}(4,2)$ representation of this group, and this we do. We then give a description of the radical 2-subgroups of $\mathfrak{A}_7$. Using these descriptions, and Propositions 1.3.5 and 4.1.12, we obtain a list of the conjugacy classes of radical 2-subgroups of $\mathbf{M_c}$ and their normalizers.

4.2. The Radical p-subgroups of $\mathrm{GL}(n,q)$

Let V be a vector space of finite dimension n over a finite field of characteristic p and order q, and let A be a radical p-subgroup of $\mathrm{GL}(\mathrm{V})$. We denote by $[\mathrm{V},\mathrm{A}^i]$ the subspace of V spanned by all commutators $[v, x_1, x_2, \ldots, x_i]$, with $i \geq 1$, $v \in \mathrm{V}$ and $x_j \in \mathrm{A}$, for $j = 1, \ldots, i$. For convenience we identify V and $[\mathrm{V},\mathrm{A}^0]$. So we have a descending chain of subspaces:

$$C : \mathrm{V} = [\mathrm{V},\mathrm{A}^0] \supset [\mathrm{V},\mathrm{A}] \supset [\mathrm{V},\mathrm{A}^2] \supset \cdots \supset [\mathrm{V},\mathrm{A}^{t-1}] \supset [\mathrm{V},\mathrm{A}^t] = 0,$$

where t is some integer between 1 and n. The normalizer $\mathrm{N}_{\mathrm{GL}(\mathrm{V})}(\mathrm{A})$ of A in $\mathrm{GL}(\mathrm{V})$ acts on each factor $[\mathrm{V},\mathrm{A}^i]/[\mathrm{V},\mathrm{A}^{i+1}]$ of C.

We let $\overline{\mathrm{A}} = \overline{\mathrm{A}}(C)$ denote the largest subgroup of $\mathrm{N}_{\mathrm{GL}(\mathrm{V})}(\mathrm{A})$ which acts trivially on each of the factors of C. Then $\overline{\mathrm{A}}$ is a p-group by [**Go80**, 5.3.3]. Also $\overline{\mathrm{A}}$ is a normal subgroup of $\mathrm{N}_{\mathrm{GL(V)}}(\mathrm{A})$. It follows from Theorem 1.3.2 that $\overline{\mathrm{A}} \leq \mathrm{A}$. But $\mathrm{A} \leq \overline{\mathrm{A}}$, by definition of $\overline{\mathrm{A}}$. So $\overline{\mathrm{A}} = \mathrm{A}$.

If $[\mathrm{V}, \mathrm{A}^i]/[\mathrm{V}, \mathrm{A}^{i+1}]$ has dimension n_{i+1}, for $i = 0,1,\ldots,t-1$, then we say A is a subgroup of GL(V) of type $(n_1, n_2, \ldots, n_t)$. Notice that $n_1 + n_2 + \cdots + n_t = n$. Clearly the above tuple of numbers uniquely determines A up to conjugacy in GL(V).

Conversely, suppose A is any p-subgroup of GL(V) of type $(n_1, n_2, \ldots, n_t)$. Then

$$\mathrm{A} = \begin{bmatrix} I_{n_1} & 0 & \cdots & 0 \\ * & I_{n_2} & \cdots & 0 \\ * & * & \ddots & \vdots \\ * & * & \cdots & I_{n_t} \end{bmatrix}, \qquad \mathrm{N}_{\mathrm{GL(V)}}(\mathrm{A}) = \begin{bmatrix} \mathrm{GL}_{n_1}(q) & 0 & \cdots & 0 \\ * & \mathrm{GL}_{n_2}(q) & \cdots & 0 \\ * & * & \ddots & \vdots \\ * & * & \cdots & \mathrm{GL}_{n_t}(q) \end{bmatrix}.$$

Hence $\mathrm{N}_{\mathrm{GL(V)}}(\mathrm{A}) \cong \mathrm{A} \rtimes (\mathrm{GL}_{n_1}(q) \times \mathrm{GL}_{n_2}(q) \times \cdots \times \mathrm{GL}_{n_t}(q))$. Since $\mathrm{GL}_m(q)$ has trivial p-core for all positive integers m, it follows that A is a radical p-subgroup of GL(V).

So we have determined the conjugacy classes of radical p-subgroups of GL(V). In fact we have shown a little more. A Sylow p-subgroup S of $\mathrm{GL}(n, q)$ is of type $(1, 1, \ldots, 1)$. The chain

$$C_{\mathrm{S}} : \mathrm{V} = [\mathrm{V}, \mathrm{S}^0] \supset [\mathrm{V}, \mathrm{S}^1] \supset [\mathrm{V}, \mathrm{S}^2] \supset \cdots \supset [\mathrm{V}, \mathrm{S}^{n-1}] \supset [\mathrm{V}, \mathrm{S}^n] = \{0\}$$

of subspaces of V is called a *flag*, since each factor is of dimension 1. Then the radical 2-subgroups of GL(V) contained in S correspond $1-1$ to the subchains of C_{S} containing both V and $\{0\}$. The one of type $(n_1, n_2, \ldots, n_t)$ corresponds to the subchain:

$$\mathrm{V} \supset [\mathrm{V}, \mathrm{S}^{n_1}] \supset [\mathrm{V}, \mathrm{S}^{n_1+n_2}] \supset \cdots \supset [\mathrm{V}, \mathrm{S}^{n_1+n_2+\cdots+n_t}] = \{0\}.$$

It follows that each radical p-subgroup of $\mathrm{GL(V)}$ contained in S is weakly closed in S with respect to $\mathrm{GL(V)}$.

We can now list representatives of the seven conjugacy classes of non-trivial radical 2-subgroups of $\mathrm{GL}(4,2)$. We let $\mathrm{A}_{i,j,\ldots,k}$, $\mathrm{Z}_{i,j,\ldots,k}$, and $\mathrm{N}_{i,j,\ldots,k}$ denote a radical 2-group of type $(i,j,\ldots,k)$, its center, and its normalizer in $\mathrm{GL}(4,2)$, respectively.

(4.2.1)

$$\mathrm{A}_{1,1,1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}, \quad \mathrm{Z}_{1,1,1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{N}_{1,1,1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}$$

$$\mathrm{A}_{2,1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}, \quad \mathrm{Z}_{2,1,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}, \quad \mathrm{N}_{2,1,1} = \begin{bmatrix} \mathrm{GL}_2(2) & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} * & * \\ * & * \end{matrix} & \begin{matrix} 1 & 0 \\ * & 1 \end{matrix} \end{bmatrix}$$

$$\mathrm{A}_{1,2,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & * & * & 1 \end{bmatrix}, \quad \mathrm{Z}_{1,2,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{N}_{1,2,1} = \begin{bmatrix} 1 & \begin{matrix} 0 & 0 \end{matrix} & 0 \\ \begin{matrix} * \\ * \end{matrix} & \mathrm{GL}_2(2) & \begin{matrix} 0 \\ 0 \end{matrix} \\ * & \begin{matrix} * & * \end{matrix} & 1 \end{bmatrix}$$

$$A_{1,1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}, \quad Z_{1,1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \end{bmatrix}, \quad N_{1,1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & \multicolumn{2}{c}{\multirow{2}{*}{GL_2(2)}} \\ * & * & & \end{bmatrix}$$

$$A_{1,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \end{bmatrix}, \quad Z_{1,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & 0 & 1 & 0 \\ * & 0 & 0 & 1 \end{bmatrix}, \quad N_{1,3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & & & \\ * & & GL_3(2) & \\ * & & & \end{bmatrix}$$

$$A_{3,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & * & * & 1 \end{bmatrix}, \quad Z_{3,1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & * & * & 1 \end{bmatrix}, \quad N_{3,1} = \begin{bmatrix} & & & 0 \\ & GL_3(2) & & 0 \\ & & & 0 \\ * & * & * & 1 \end{bmatrix}$$

$$A_{2,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}, \quad Z_{2,2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{bmatrix}, \quad N_{2,2} = \begin{bmatrix} \multicolumn{2}{c}{\multirow{2}{*}{GL_2(2)}} & 0 & 0 \\ & & 0 & 0 \\ * & * & \multicolumn{2}{c}{\multirow{2}{*}{GL_2(2)}} \\ * & * & & \end{bmatrix}$$

We immediately obtain the following inclusions among the radical 2-subgroups of $\mathrm{GL}(4,2)$ and their normalizers:

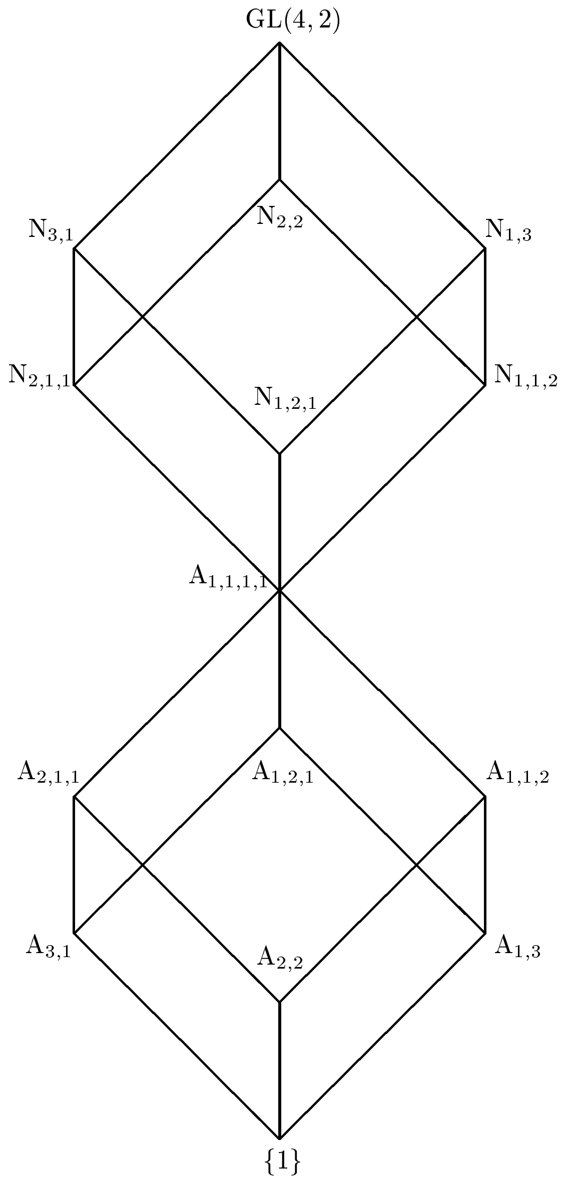

FIGURE 1. The radical 2-subgroups of $\mathrm{GL}(4,2)$ and their normalizers in $\mathrm{GL}(4,2)$

τ	a fixed involution of $\mathbf{M_c}$
$\mathbf{H} \cong 2.\mathfrak{A}_8$	the centralizer of τ in $\mathbf{M_c}$
$\mathbf{H}.2 \cong 2.\mathfrak{S}_8$	the centralizer of τ in $\mathbf{M_c}.2$
$\mathbf{T}$	the group $\langle \tau \rangle = \mathrm{Z}(\mathbf{H})$
$\mathbf{R}_{1,1,1,1}$	a fixed Sylow 2-subgroup of $\mathbf{H}$
$\mathrm{A}_{i,j,\dots,k}$	the unique radical 2-subgroup of $\mathbf{H}/\mathbf{T}$ of type
	$(i, j, \dots, k)$ contained in $\mathbf{R}_{1,1,1,1}/\mathbf{T}$
$\mathrm{Z}_{i,j,\dots,k}$	the center of $\mathrm{A}_{i,j,\dots,k}$
$\mathrm{N}_{i,j,\dots,k}$	the normalizer of $\mathrm{A}_{i,j,\dots,k}$ in $\mathbf{H}/\mathbf{T}$
$\mathbf{R}_{i,j,\dots,k}$	the covering group of $\mathrm{A}_{i,j,\dots,k}$ in $\mathbf{H}$
ε	the generator of the center of $\mathrm{A}_{1,1,1,1}$
$\tilde{\varepsilon}$	an element of $\mathbf{H}$ whose image in $\mathbf{H}/\mathbf{T}$ is ε
$\Lambda_{3,1}$	a fixed complement to $\mathbf{R}_{3,1}$ in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) \cong \mathbf{R}_{3,1} \rtimes \mathfrak{A}_7$
$\Lambda_{1,3}$	a fixed complement to $\mathbf{R}_{1,3}$ in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3}) \cong \mathbf{R}_{1,3} \rtimes \mathfrak{A}_7$
$\mathrm{G}_{3,1}$	a fixed complement to $\mathbf{R}_{3,1}$ in $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}) \cong \mathbf{R}_{3,1} \rtimes \mathrm{GL}(3,2)$
$\mathrm{G}_{1,3}$	a fixed complement to $\mathbf{R}_{1,3}$ in $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,3}) \cong \mathbf{R}_{1,3} \rtimes \mathrm{GL}(3,2)$

TABLE 4.1. Notation for Elements and Subgroups

4.3. The Radical 2-subgroups of H

Table 4.1 summarises the notation we shall use for the rest of Chapter 4.

We remark that from (4.2.1) the center $\mathrm{Z}_{1,1,1,1}$ of $\mathrm{A}_{1,1,1,1} = \mathbf{R}_{1,1,1,1}/\mathbf{T}$ is cyclic of order 2. Let ε denote a generator of this group, as specified in Table 4.1. Then the order of the centralizer of ε in $\mathbf{H}/\mathbf{T} \cong \mathfrak{A}_8$ is divisible by $|\mathrm{A}_{1,1,1,1}| = 2^6$. From the orders of the centralizers of elements of $\mathfrak{A}_8$, it is clear that ε comes from the class $(2A)$ of $\mathfrak{A}_8$.

We now investigate the radical 2-subgroups of $\mathbf{H}$.

LEMMA 4.3.1. $\mathrm{Z}(\mathbf{R}_{1,1,1,1}) = \mathbf{T}$.

PROOF. Let $\tilde{\varepsilon}$ denote an element of $\mathbf{H}$ whose image in $\mathbf{H}/\mathbf{T}$ is ε. Then $\mathrm{Z}(\mathbf{R}_{1,1,1,1}) \leq \langle \tau, \tilde{\varepsilon} \rangle$. But from the Atlas the class $(2A)$ does not split in $\mathbf{H}$. So $\tilde{\varepsilon} \notin \mathrm{Z}(\mathbf{R}_{1,1,1,1})$, and the latter group must coincide with $\mathbf{T}$. ☐

LEMMA 4.3.2. $\mathrm{Z}(\mathbf{R}_{2,1,1}) = \mathbf{T}$ *and* $\mathrm{Z}(\mathbf{R}_{1,1,2}) = \mathbf{T}$.

PROOF. From (4.2.1) the center $\mathrm{Z}_{2,1,1}$ of $\mathrm{A}_{2,1,1}$ is an elementary abelian group of order 2^2 which properly contains $\langle \varepsilon \rangle$. Moreover the normalizer $\mathrm{N}_{2,1,1}$ acts on $\mathrm{Z}_{2,1,1}$ as the full automorphism group $\mathrm{Aut}(\mathrm{Z}_{2,1,1}) \cong \mathrm{GL}(2,2)$. It follows that $\mathrm{Z}(\mathbf{R}_{2,1,1})$ is either $\mathbf{T}$ or the full inverse image $\tilde{\mathrm{Z}}_{2,1,1}$ of $\mathrm{Z}_{2,1,1}$ in $\mathbf{H}$. Similarly $\mathrm{Z}(\mathbf{R}_{1,1,2})$ is either $\mathbf{T}$ or the full inverse image $\tilde{\mathrm{Z}}_{1,1,2}$ of $\mathrm{Z}_{1,1,2}$ in $\mathbf{H}$.

Suppose $\mathrm{Z}(\mathbf{R}_{2,1,1}) = \tilde{\mathrm{Z}}_{2,1,1}$. Then $\mathrm{Z}(\mathbf{R}_{1,1,2}) = \tilde{\mathrm{Z}}_{1,1,2}$, since $\mathbf{R}_{2,1,1}$ and $\mathbf{R}_{1,1,2}$ are conjugate in $\mathbf{H}.2$, and in particular their centers are isomorphic. Hence $\tilde{\varepsilon} \in \tilde{\mathrm{Z}}_{2,1,1} \cap \tilde{\mathrm{Z}}_{1,1,2}$ centralizes $\mathbf{R}_{2,1,1}\mathbf{R}_{1,1,2} = \mathbf{R}_{1,1,1,1}$. This contradicts the fact that $\mathrm{Z}(\mathbf{R}_{1,1,1,1}) = \mathbf{T}$. We conclude that $\mathrm{Z}(\mathbf{R}_{2,1,1}) = \mathbf{T}$ and $\mathrm{Z}(\mathbf{R}_{1,1,2}) = \mathbf{T}$. ☐

COROLLARY 4.3.3. *The normalizer in* $\mathbf{M_c}$ *of each of the groups* $\mathbf{R}_{1,1,1,1}, \mathbf{R}_{2,1,1}$ *and* $\mathbf{R}_{1,1,2}$ *is contained in* $\mathbf{H}$, *and the normalizer in* $\mathbf{M_c}.2$ *is contained in* $\mathbf{H}.2$. *In particular, each of these groups is a radical* 2*-subgroup of* $\mathbf{M_c}$.

PROOF. Each of the groups $\mathbf{R}_{1,1,1,1}, \mathbf{R}_{2,1,1}$ and $\mathbf{R}_{1,1,2}$ is radical in $\mathbf{H}$. So the result follows at once from Lemmas 4.3.1 and 4.3.2, and the fact that $\mathbf{H} = \mathrm{C}_{\mathbf{M}_c}(\mathbf{T})$. $\square$

It turns out to be useful to represent $\mathrm{A}_{2,2}$ as a subgroup of $\mathfrak{A}_8$ in order to describe its covering group $\mathbf{R}_{2,2}$ in $\mathbf{H}$. So we first need some results on the elementary abelian 2-subgroups of $\mathfrak{A}_8$.

LEMMA 4.3.4. *Suppose* V *is a finite set and* X *is a maximal elementary abelian* p*-subgroup of* $\mathfrak{S}(\mathrm{V})$*. Then there exist decompositions:*

$$\begin{aligned} \mathrm{V} &= \mathrm{V}_0 \cup \mathrm{V}_1 \cup \mathrm{V}_2 \cup \cdots \cup \mathrm{V}_t, \\ \mathrm{X} &= \mathrm{X}_1 \times \mathrm{X}_2 \times \cdots \times \mathrm{X}_t, \end{aligned} \tag{4.3.5}$$

of V *and* X*, where* X *fixes each element of* V_0*, the* V_i *are pairwise disjoint subsets of* $\mathrm{V}\backslash\mathrm{V}_0$*, and* X_i *is a non-trivial elementary abelian* p*-subgroup of* $\mathfrak{S}(\mathrm{V}_i)$ *acting regularly and transitively on* V_i*, for* $i = 1, 2, \ldots, t$*.*

PROOF. Let V_0 be the set of fixed points, and $\mathrm{V}_1, \mathrm{V}_2, \ldots, \mathrm{V}_t$ be the distinct X-orbits of length > 1 on V. We identify $\mathfrak{S}(\mathrm{V}_i)$, for $i = 1, 2, \ldots, t$, with the subgroup of all permutations in $\mathfrak{S}(\mathrm{V})$ fixing each element in $\mathrm{V}\backslash\mathrm{V}_i$. Hence $\mathfrak{S}(\mathrm{V}_1) \times \mathfrak{S}(\mathrm{V}_2) \times \cdots \times \mathfrak{S}(\mathrm{V}_t)$ is a subgroup of $\mathfrak{S}(\mathrm{V})$. Let X_i be the image of X under restriction to V_i, for each $i = 1, 2, \ldots, t$. Each X_i is a regular transitive elementary abelian p-subgroup of $\mathfrak{S}(\mathrm{V}_i)$. Also $\mathrm{X} \leq \mathrm{X}_1 \times \mathrm{X}_2 \times \cdots \times \mathrm{X}_t$. By the maximality of X, this inequality is actually an equality. $\square$

We list the maximal elementary abelian 2-subgroups of $\mathfrak{S}_8$ to within conjugacy in $\mathfrak{S}_8$. Since 8 is even, the number of trivial orbits of any 2-subgroup is even. So no maximal elementary abelian 2-subgroup of $\mathfrak{S}_8$ has any trivial orbits.

Orbit Lengths	Elementary Abelian 2-group	Intersection with $\mathfrak{A}_8$
(8)	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
$(4,4)$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$
$(4,2,2)$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
$(2,2,2,2)$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

Using the above table, it is clear that $\mathfrak{S}_8$ has a unique conjugacy class $\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$ of elementary abelian 2-subgroups of order $\geq 2^4$ contained in $\mathfrak{A}_8$. Since the normalizer in $\mathfrak{S}_8$ of such a subgroup is not contained in $\mathfrak{A}_8$, we see that $\mathfrak{A}_8$ also has a unique conjugacy class of such subgroups. Clearly $\mathrm{A}_{2,2}$ is of this form. So we may choose an identification of $\mathrm{GL}(4,2)$ with $\mathfrak{A}_8$ so that

$$\mathrm{A}_{2,2} = \big\langle (1,2)(3,4), (1,3)(2,4) \big\rangle \times \big\langle (5,6)(7,8), (5,7)(6,8) \big\rangle. \tag{4.3.6}$$

LEMMA 4.3.7. *Let* K *be any four subgroup of* $\mathfrak{A}_8$ *whose nontrivial elements come from the class* $(2B)$ *of involutions of type* $(..)(..)$. *Then the covering group of* K *in* $\mathbf{H} = 2.\mathfrak{A}_8$ *is isomorphic to a quaternion group* Q_8 *of order* 8.

PROOF. From the Atlas, the inverse images of elements of type $(i,j)(k,l)$ are of order 4 in $\mathbf{H}$. Thus the covering group of K has a unique involution, namely τ. Since the covering group is non-cyclic of order 8, it must be isomorphic to Q_8. $\square$

LEMMA 4.3.8. *Let $\sigma \in \mathfrak{S}_8$, and suppose x_1 and x_2 are elements of $\mathbf{H}$ whose images in $\mathbf{H}/\mathbf{T} \cong \mathfrak{A}_8$ are $\big(\sigma(1),\sigma(2)\big)\big(\sigma(3),\sigma(4)\big)$ and $\big(\sigma(5),\sigma(6)\big)\big(\sigma(7),\sigma(8)\big)$, respectively. Then $[x_1, x_2] = 1$.*

PROOF. Let x_3 and x_4 be elements of $\mathbf{H}$ having images $\big(\sigma(1),\sigma(2)\big)\big(\sigma(5),\sigma(6)\big)$ and $\big(\sigma(1),\sigma(2)\big)\big(\sigma(7),\sigma(8)\big)$, respectively in $\mathfrak{A}_8$. By Lemma 4.3.7 the groups $\langle x_1, x_3\rangle$ and $\langle x_1, x_4\rangle$ are quaternion. So $[x_1, x_3] = [x_1, x_4] = \tau$. Then $[x_1, x_3x_4] = [x_1, x_4][x_1, x_3]^{x_4} = \tau^2 = 1$. But $x_2 \equiv x_3x_4$ modulo $\langle\tau\rangle$. The result now follows. $\square$

We can now prove the following

LEMMA 4.3.9. *The covering group $\mathbf{R}_{2,2}$ of $\mathrm{A}_{2,2}$ in $\mathbf{H}$ is extra-special of order 2^5. In particular, its center is generated by τ. So $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{2,2}) \leq \mathbf{H}$ and $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{2,2}) \leq \mathbf{H}.2$. Hence $\mathbf{R}_{2,2}$ is a radical 2-subgroup of $\mathbf{M_c}$.*

PROOF. By (4.3.6) the group $\mathrm{A}_{2,2}$ is the direct product of two regular elementary abelian 2-groups of type $\mathbb{Z}_2^2$. By Lemma 4.3.7 the covering group of each is quaternion, and by Lemma 4.3.8 the two covering groups commute. Hence $\mathbf{R}_{2,2}$ is the central product of two Q_8's and $\mathrm{Z}(\mathbf{R}_{2,2}) = \mathbf{T}$.

Finally $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{2,2}) \leq \mathbf{H}$, since $\mathbf{H} = \mathrm{N}_{\mathbf{M_c}}(\mathbf{T})$, and $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{2,2}) \leq \mathbf{H}.2$, since $\mathbf{H}.2 = \mathrm{N}_{\mathbf{M_c}.2}(\mathbf{T})$. $\square$

Next we investigate the covering groups $\mathbf{R}_{3,1}$ and $\mathbf{R}_{1,3}$ of $\mathrm{A}_{3,1}$ and $\mathrm{A}_{1,3}$ in $\mathbf{H}$.

LEMMA 4.3.10. *The groups $\mathbf{R}_{3,1}$ and $\mathbf{R}_{1,3}$ are representatives of the two classes of maximal elementary abelian 2-subgroups of $\mathbf{M_c}$ of order 2^4. In particular they are conjugate in $\mathbf{M_c}.2$.*

Proof. From (4.2.1) the normalizer $\mathrm{N}_{3,1}$ of $\mathrm{A}_{3,1}$ is isomorphic to the holomorph $\mathrm{A}_{3,1} \rtimes \mathrm{GL}(3,2)$ of $\mathrm{A}_{3,1}$. Hence all elements of $\mathrm{A}_{3,1}^{\#}$ are conjugate in $\mathrm{N}_{3,1}$. As remarked before Lemma 4.3.1, the element $\varepsilon \in \mathrm{A}_{3,1}$ comes from the $(2A)$ conjugacy class of $\mathfrak{A}_8$. So all elements of $\mathrm{A}_{3,1}^{\#}$ come from this class also. Then from Lemma 4.1.2, all elements of $\mathbf{R}_{3,1}\backslash\mathbf{T}$ come from the single class of non-central involutions of $\mathbf{H}$. Hence $\mathbf{R}_{3,1}$ has exponent 2. We conclude that it is in one of the two classes of maximal elementary abelian 2-subgroups of $\mathbf{H}$ of order 2^4 given by Lemma 4.1.7.

A similar statement can be made for $\mathbf{R}_{1,3}$.

Since $\mathrm{A}_{3,1}$ and $\mathrm{A}_{1,3}$ are not conjugate in $\mathbf{H}/\mathbf{T}$, their inverse images $\mathbf{R}_{3,1}$ and $\mathbf{R}_{1,3}$ cannot be conjugate in $\mathbf{H}$. The lemma now follows from Corollary 4.1.9. □

Corollary 4.3.11. *$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) = \mathbf{R}_{3,1} \rtimes \Lambda_{3,1}$, where $\Lambda_{3,1} \cong \mathfrak{A}_7$ acts flag transitively on $\mathbf{R}_{3,1}$. Hence $\mathbf{R}_{3,1}$ is a radical 2-subgroup of $\mathbf{M_c}$.*

Proof. The first statement follows from Lemmas 4.1.8 and 4.3.10. Since $\mathfrak{A}_7$ is simple, $\mathbf{R}_{3,1}$ is the 2-core of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$. This proves the second statement. □

Similar statements can be made about $\mathbf{R}_{1,3}$ and its normalizer $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3}) = \mathbf{R}_{1,3} \rtimes \Lambda_{1,3}$.

Corollary 4.3.12. *The groups $\mathbf{R}_{3,1}$ and $\mathbf{R}_{1,3}$ are the unique subgroups of $\mathbf{R}_{1,1,1,1}$ isomorphic to $\mathbb{Z}_2^4$. In particular both are weakly closed in $\mathbf{R}_{1,1,1,1}$ with respect to $\mathbf{M_c}$.*

Proof. This is an immediate consequence of the lemma above and Corollary 4.1.10. □

Next we take a look at the covering group $\mathbf{R}_{1,2,1}$ of $\mathrm{A}_{1,2,1}$ in $\mathbf{H}$. The first result is easy.

LEMMA 4.3.13. $\mathbf{R}_{1,2,1} = \mathbf{R}_{3,1}\mathbf{R}_{1,3}$.

PROOF. This follows at once from (4.2.1). $\square$

LEMMA 4.3.14. $\mathrm{Z}(\mathbf{R}_{1,2,1}) = \mathrm{C}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) = \langle \tau, \tilde{\varepsilon} \rangle \cong \mathbb{Z}_2^2$. *Hence* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) = \mathrm{N}_{\mathbf{M_c}}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big) = \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) \cap \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3})$. *Moreover* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ *acts as the full automorphism group on* $\mathrm{Z}(\mathbf{R}_{1,2,1})$.

PROOF. By Lemmas 4.1.5 and 4.3.10 the groups $\mathbf{R}_{3,1}$ and $\mathbf{R}_{1,3}$ are self-centralizing in $\mathbf{M_c}$. Hence $\mathrm{C}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) = \mathrm{C}_{\mathbf{M_c}}(\mathbf{R}_{3,1}\mathbf{R}_{1,3}) = \mathbf{R}_{3,1} \cap \mathbf{R}_{1,3} = \langle \tau, \tilde{\varepsilon} \rangle$. So $\mathrm{Z}(\mathbf{R}_{1,2,1}) = \langle \tau, \tilde{\varepsilon} \rangle$.

The group $\mathrm{Z}(\mathbf{R}_{1,2,1}) = \mathbf{R}_{3,1} \cap \mathbf{R}_{1,3}$ strictly contains $\mathbf{T}$. It then follows from Corollary 4.1.11 that $\mathrm{N}_{\mathbf{M_c}}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big) \leq \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) \cap \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3})$. But $\mathbf{R}_{1,2,1} = \mathbf{R}_{3,1}\mathbf{R}_{1,3}$. So $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) \geq \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) \cap \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3})$. Hence $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) \leq \mathrm{N}_{\mathbf{M_c}}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big) \leq \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) \cap \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3}) \leq \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$. So these inequalities are all equalities.

From Corollary 4.3.11 the group $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$ acts flag transitively on $\mathbf{R}_{3,1}$. Hence the normalizer of $\mathrm{Z}(\mathbf{R}_{1,2,1}) = \langle \tau, \tilde{\varepsilon} \rangle$ in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$ acts as the full automorphism group on $\mathrm{Z}(\mathbf{R}_{1,2,1}) \cong \mathbb{Z}_2^2$. But this normalizer is $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$. This completes the lemma. $\square$

COROLLARY 4.3.15. $[\mathbf{R}_{1,2,1}, \mathbf{R}_{1,2,1}] = \Phi(\mathbf{R}_{1,2,1}) = \mathrm{Z}(\mathbf{R}_{1,2,1})$. *If* $\mathbf{T}^{(1)}$ *is any subgroup of* $\mathrm{Z}(\mathbf{R}_{1,2,1})$ *of order* 2*, then* $\mathbf{R}_{1,2,1}/\mathbf{T}^{(1)}$ *is extra-special of order* 2^5.

PROOF. Since $\mathbf{R}_{1,2,1}/\mathrm{Z}(\mathbf{R}_{1,2,1}) \cong \mathrm{A}_{1,2,1}/\mathrm{Z}(\mathrm{A}_{1,2,1})$ is an elementary abelian 2-group, it follows that $[\mathbf{R}_{1,2,1}, \mathbf{R}_{1,2,1}]$ and $\Phi(\mathbf{R}_{1,2,1})$ are non-trivial $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$-invariant subgroups of $\mathrm{Z}(\mathbf{R}_{1,2,1})$. But $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ acts as the full automorphism group on $\mathrm{Z}(\mathbf{R}_{1,2,1})$, by Lemma 4.3.14. Hence $[\mathbf{R}_{1,2,1}, \mathbf{R}_{1,2,1}] = \Phi(\mathbf{R}_{1,2,1}) = \mathrm{Z}(\mathbf{R}_{1,2,1})$.

It also follows from the action of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ on $\mathrm{Z}(\mathbf{R}_{1,2,1})$ that $\mathbf{R}_{1,2,1}/\mathbf{T}^{(1)} \cong \mathbf{R}_{1,2,1}/\mathbf{T}$. Since this latter group is extra-special of order 2^5, this completes the proof of the lemma. $\square$

4.4. The Radical 2-chains of $\mathrm{M_c}$

LEMMA 4.4.1. *A Sylow* 2*-subgroup* B_1 *of* $\mathfrak{A}_7$ *is isomorphic to* D_8*. So it has two non-cyclic subgroups* B_2 *and* B_3 *of order* 4*, each isomorphic to* $\mathbb{Z}_2^2$*. They can be ordered so that* $\mathrm{N}_{\mathfrak{A}_7}(\mathrm{B}_2)$ *is isomorphic to* $\mathfrak{S}_4$*, while* $\mathrm{N}_{\mathfrak{A}_7}(\mathrm{B}_3)$ *has the form* $(\mathfrak{A}_4 \times 3)\colon 2$*. Both* B_2 *and* B_3 *are radical* 2*-subgroups of* $\mathfrak{A}_7$*. So too are* B_1*, which is self-normalizing in* $\mathfrak{A}_7$*, and* $\mathrm{B}_4 = \{1\}$*, whose normalizer is* $\mathfrak{A}_7$*. Any radical* 2*-subgroup of* $\mathfrak{A}_7$ *is conjugate to exactly one of* B_1, B_2, B_3, B_4.

PROOF. Let $\mathrm{B}_1 = \langle (1,3,2,4)(5,6), (1,2)(5,6) \rangle$ be a fixed Sylow 2-subgroup of $\mathfrak{A}_7$. Then B_1 is dihedral of order 8. Also B_1 is self-normalizing in $\mathfrak{A}_7$. Hence it is a radical 2-subgroup of $\mathfrak{A}_7$.

The trivial group $\mathrm{B}_4 = \{1\}$ is a radical 2-subgroup of $\mathfrak{A}_7$, since it is the 2-core of its normalizer $\mathfrak{A}_7$.

The centralizer of any involution of $\mathfrak{A}_7$ is isomorphic to $\left(\mathbb{Z}_2^2 \times \mathbb{Z}_3\right) : 2$. Hence there are no radical 2-subgroups of $\mathfrak{A}_7$ of order 2.

Any radical 2-subgroup of $\mathfrak{A}_7$ of order 4 is conjugate to one of the three subgroups of order 4 in $\mathrm{B}_1 \cong D_8$.

The group $\langle(1,3,2,4)(5,6)\rangle$ is self-centralizing in $\mathfrak{A}_7$. Hence its normalizer is B_1. So this group is not radical in $\mathfrak{A}_7$.

Let $\mathrm{B}_2 = \langle(1,2)(3,4),(1,2)(5,6)\rangle$. Then $\mathrm{N}_{\mathfrak{A}_7}(\mathrm{B}_2) \cong \mathfrak{S}_4$. So B_2 is a radical 2-subgroup of $\mathfrak{A}_7$.

We set $\mathrm{B}_3 = \langle(1,2)(3,4),(1,3)(2,4)\rangle$. Its normalizer in $\mathfrak{S}_7$ is isomorphic to $\mathfrak{S}_4 \times \mathfrak{S}_3$. Thus $\mathrm{N}_3 = \big(\langle \mathrm{B}_3,(1,2,3)\rangle \times \langle(5,6,7)\rangle\big) \rtimes \langle(1,2)(5,6)\rangle \cong (\mathfrak{A}_4 \times 3) : 2$ is the normalizer of B_3 in $\mathfrak{A}_7$. So B_3 is also radical in $\mathfrak{A}_7$. $\square$

Recall from Corollary 4.3.11 and Table 4.1 on page 97 that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) = \mathbf{R}_{3,1} \rtimes \Lambda_{3,1}$, where $\Lambda_{3,1} \cong \mathfrak{A}_7$. We identify $\Lambda_{3,1}$ with $\mathfrak{A}_7$ in such a way that the Sylow 2-subgroup B_1 of $\Lambda_{3,1} = \mathfrak{A}_7$ is $\mathbf{R}_{1,1,1,1} \cap \Lambda_{3,1}$. So B_1, B_2 and B_3 are radical 2-subgroups of $\Lambda_{3,1}$ contained in $\mathbf{R}_{1,1,1,1}$.

LEMMA 4.4.2. $\mathbf{R}_{3,1} \rtimes \mathrm{B}_1 = \mathbf{R}_{1,1,1,1}$, *while* $\mathbf{R}_{3,1} \rtimes \mathrm{B}_2 = \mathbf{R}_{2,1,1}$ *and* $\mathbf{R}_{3,1} \rtimes \mathrm{B}_3 = \mathbf{R}_{1,2,1}$. *Hence* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) = \mathrm{N}_{\mathbf{R}_{3,1}\rtimes\Lambda_{3,1}}(\mathbf{R}_{1,2,1}) = \mathbf{R}_{3,1} \rtimes \mathrm{N}_{\Lambda_{3,1}}(\mathrm{B}_3) \cong \mathbf{R}_{3,1} \rtimes \big((\mathfrak{A}_4 \times 3) : 2\big) \cong \mathbf{R}_{1,2,1} \rtimes (3^2 : 2)$. *In particular* $\mathbf{R}_{1,2,1}$ *is a radical* 2*-subgroup of* $\mathbf{M_c}$.

PROOF. From the group orders, $\mathbf{R}_{3,1} \rtimes \mathrm{B}_1 = \mathbf{R}_{1,1,1,1}$.

Since $\mathbf{R}_{3,1} \leq \mathbf{R}_{2,1,1} \leq \mathbf{R}_{3,1} \rtimes \Lambda_{3,1}$, the group $\mathbf{R}_{2,1,1} \cap \Lambda_{3,1}$ is a maximal subgroup of B_1 isomorphic to $\mathbb{Z}_2^2$. Thus $\mathbf{R}_{2,1,1} \cap \Lambda_{3,1}$ is a radical 2-subgroup of $\Lambda_{3,1}$. Also $\mathrm{N}_{\mathbf{R}_{3,1}\rtimes\Lambda_{3,1}}(\mathbf{R}_{2,1,1}) = \mathbf{R}_{3,1} \rtimes \big(\mathrm{N}_{\Lambda_{3,1}}(\mathbf{R}_{2,1,1} \cap \Lambda_{3,1})\big)$. But by Equations (4.2.1) and Corollary 4.3.1 the group $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{2,1,1}) = \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{2,1,1}) = 2.\mathrm{N}_{2,1,1}$ has order $2^7 \cdot 3$. Hence $|\mathrm{N}_{\Lambda_{3,1}}(\mathbf{R}_{2,1,1} \cap \Lambda_{3,1})|$ has order $\leq 2^3 \cdot 3$. We conclude from this and Lemma 4.4.1 that $\mathbf{R}_{2,1,1} \cap \Lambda_{3,1} = \mathrm{B}_2$.

Similarly $\mathbf{R}_{1,2,1} \cap \Lambda_{3,1}$ is a maximal non-cyclic subgroup of B_1 which is different from $\mathbf{R}_{2,1,1} \cap \Lambda_{3,1} = \mathrm{B}_2$. Hence $\mathbf{R}_{1,2,1} \cap \Lambda_{3,1} = \mathrm{B}_3$.

From Lemma 4.3.14, the normalizer $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ of $\mathbf{R}_{1,2,1}$ in $\mathbf{M_c}$ is the intersection $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) \cap \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3})$ of the normalizers of $\mathbf{R}_{3,1}$ and $\mathbf{R}_{1,3}$. In particular $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) = \mathrm{N}_{\mathbf{R}_{3,1} \rtimes \Lambda_{3,1}}(\mathbf{R}_{1,2,1}) = \mathbf{R}_{3,1} \rtimes \mathrm{N}_{\Lambda_{3,1}}(\mathrm{B}_3)$. The description of this last group comes from Lemma 4.4.1. □

We can now prove the following

Proposition 4.4.3. $\mathbf{M_c}$ *has nine conjugacy classes of radical* 2*-subgroups, with representatives given in Table 4.2 on the following page.*

Proof. This follows from Proposition 4.1.12, Equations (4.2.1), Corollaries 4.3.3 and 4.3.11, and Lemmas 4.3.9 and 4.4.2. The radical 2-subgroups $\mathbf{R}_{3,1}$ and $\mathbf{R}_{1,3}$ are conjugate in $\mathbf{H}.2$, and hence in $\mathbf{M_c}.2$ by Lemma 4.3.10. Then from Figure 1 on page 96, so too are $\mathbf{R}_{2,1,1}$ and $\mathbf{R}_{1,1,2}$. No other pairs of radical 2-subgroups in Table 4.2 on the following page are isomorphic as groups. So $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R})$ is some extension $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}).2$ of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R})$, when $\mathbf{R}$ is one of $\mathbf{R}_{1,1,1,1}$, $\mathbf{R}_{1,2,1}$, $\mathbf{R}_{2,2}$, $\mathbf{T}$ or $\{1\}$. For reasons outlined in Proposition 4.4.9 below, we do not need detailed information about the normalizers of $\mathbf{R}_{1,1,1,1}$, $\mathbf{R}_{2,1,1}$, $\mathbf{R}_{1,1,2}$ or $\mathbf{R}_{2,2}$ in either $\mathbf{M_c}$ or $\mathbf{M_c}.2$. We have already seen that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{T}) = \mathbf{H}.2 \cong 2.\mathfrak{S}_8$. Clearly $\mathrm{N}_{\mathbf{M_c}.2}(\{1\}) = \mathbf{M_c}.2$. The structure of $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})$ is investigated in Section 4.9 below. □

We will also need a list of the conjugacy classes of radical 2-subgroups of $\mathrm{GL}(3,2)$. The notation and methods are as in Section 4.2.

Radical 2-subgroup $\mathbf{R}$	Structure	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R})$	$\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R})$
$\mathbf{R}_{1,1,1,1}$	$\mathbb{Z}_2^4 \rtimes D_8$	$\mathbf{R}_{1,1,1,1}$	$\mathbf{R}_{1,1,1,1}.2$
$\mathbf{R}_{2,1,1}$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2^2$	$\mathbf{R}_{2,1,1} \rtimes \mathfrak{S}_3$	$\mathbf{R}_{2,1,1} \rtimes \mathfrak{S}_3$
$\mathbf{R}_{1,1,2}$	$\mathbb{Z}_2^4 \rtimes \mathbb{Z}_2^2$	$\mathbf{R}_{1,1,2} \rtimes \mathfrak{S}_3$	$\mathbf{R}_{1,1,2} \rtimes \mathfrak{S}_3$
$\mathbf{R}_{3,1}$	$\mathbb{Z}_2^4$	$\mathbf{R}_{3,1} \rtimes \Lambda_{3,1}$	$\mathbf{R}_{3,1} \rtimes \Lambda_{3,1}$
$\mathbf{R}_{1,3}$	$\mathbb{Z}_2^4$	$\mathbf{R}_{1,3} \rtimes \Lambda_{1,3}$	$\mathbf{R}_{1,3} \rtimes \Lambda_{1,3}$
$\mathbf{R}_{1,2,1}$	$\mathbb{Z}_2^2.\mathbb{Z}_2^4$	$\mathbf{R}_{1,2,1} \rtimes (3^2 : 2)$	$\mathbf{R}_{1,2,1} \rtimes (\mathfrak{S}_3 \times \mathfrak{S}_3)$
$\mathbf{R}_{2,2}$	2^{1+4}	$\mathbf{R}_{2,2}.(\mathfrak{S}_3 \times \mathfrak{S}_3)$	$\mathbf{R}_{2,2}.(\mathfrak{S}_3 \times \mathfrak{S}_3).2$
$\mathbf{T}$	$\mathbb{Z}_2$	$\mathbf{H} \cong 2.\mathfrak{A}_8$	$\mathbf{H}.2 \cong 2.\mathfrak{S}_8$
$\{1\}$	$\{1\}$	$\mathbf{M_c}$	$\mathbf{M_c}.2$

TABLE 4.2. The Radical 2-subgroups of $\mathbf{M_c}$

LEMMA 4.4.4. *There are four conjugacy classes of radical 2-subgroups of* $\mathrm{GL}(3,2)$*, represented by:*

$$\mathrm{A}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{A}_{1,2} = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & 0 & 1 \end{bmatrix}, \quad \mathrm{A}_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & * & 1 \end{bmatrix}, \quad \mathrm{A}_{1,1,1} = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}.$$

Their normalizers in $\mathrm{GL}(3,2)$ *are* $\mathrm{N}_3 = \mathrm{GL}(3,2)$, $\mathrm{N}_{1,2} \cong \mathrm{A}_{1,2} \rtimes \mathrm{GL}(2,2) \cong \mathfrak{S}_4$, $\mathrm{N}_{2,1} \cong \mathrm{A}_{2,1} \rtimes \mathrm{GL}(2,2) \cong \mathfrak{S}_4$*, and* $\mathrm{N}_{1,1,1} = \mathrm{A}_{1,1,1} \cong D_8$ *respectively.*

PROOF. This follows directly from the results of Section 4.2. $\square$

Recall from Lemma 4.1.5 and Table 4.1 on page 97 that $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}) = \mathbf{R}_{3,1} \rtimes \mathrm{G}_{3,1}$, where $\mathrm{G}_{3,1}$ is isomorphic to $\mathrm{GL}(3,2)$. We identify $\mathrm{G}_{3,1}$ and $\mathrm{GL}(3,2)$ in such a way that the Sylow 2-subgroup $\mathrm{A}_{1,1,1}$ of $\mathrm{G}_{3,1} = \mathrm{GL}(3,2)$ corresponds to $\mathbf{R}_{1,1,1,1} \cap \mathrm{G}_{3,1}$.

LEMMA 4.4.5. *Every radical 2-subgroup of* $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}) = \mathbf{R}_{3,1} \rtimes \mathrm{G}_{3,1}$ *is conjugate in* $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$ *to one of the groups* $\mathbf{R}_{3,1}$, $\mathbf{R}_{2,1,1}$, $\mathbf{R}_{1,2,1}$ *or* $\mathbf{R}_{1,1,1,1}$. *Similarly every radical 2-subgroup of* $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,3}) = \mathbf{R}_{1,3} \rtimes \mathrm{G}_{1,3}$ *is conjugate in* $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,3})$ *to one of the groups* $\mathbf{R}_{1,3}$, $\mathbf{R}_{1,1,2}$, $\mathbf{R}_{1,2,1}$ *or* $\mathbf{R}_{1,1,1,1}$.

PROOF. There are 4 conjugacy classes of radical 2-subgroups of $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}) = \mathbf{R}_{3,1} \rtimes \mathrm{G}_{3,1}$. These classes are represented by $\mathbf{R}_{3,1} \rtimes \mathrm{A}_3$, $\mathbf{R}_{3,1} \rtimes \mathrm{A}_{1,2}$, $\mathbf{R}_{3,1} \rtimes \mathrm{A}_{2,1}$, and $\mathbf{R}_{3,1} \rtimes \mathrm{A}_{1,1,1}$, using the notation of Lemma 4.4.4. It is easy to see that $\mathbf{R}_{3,1} \rtimes \mathrm{A}_3 = \mathbf{R}_{3,1}$ and $\mathbf{R}_{3,1} \rtimes \mathrm{A}_{1,1,1} = \mathbf{R}_{1,1,1,1}$, while $\mathbf{R}_{3,1} \rtimes \mathrm{A}_{1,2}$ and $\mathbf{R}_{3,1} \rtimes \mathrm{A}_{2,1}$ are $\mathbf{R}_{2,1,1}$ and $\mathbf{R}_{1,2,1}$ in some order. $\square$

We now deal with the radical 2-subgroups of of $\mathrm{N}_{\mathbf{M}_c}(\mathbf{R}_{2,2})$.

LEMMA 4.4.6. *Every radical 2-subgroup of* $\mathrm{N}_{\mathbf{M}_c}(\mathbf{R}_{2,2})$ *is conjugate in* $\mathrm{N}_{\mathbf{M}_c}(\mathbf{R}_{2,2})$ *to one of the groups* $\mathbf{R}_{2,2}$, $\mathbf{R}_{2,1,1}$, $\mathbf{R}_{1,1,2}$ *or* $\mathbf{R}_{1,1,1,1}$.

PROOF. The factor group $\mathrm{N}_{\mathbf{M}_c}(\mathbf{R}_{2,2})/\mathbf{R}_{2,2}$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_3$. There are four conjugacy classes of radical 2-subgroups in this factor group, represented by 1×1, $\mathfrak{S}_2 \times 1$, $1 \times \mathfrak{S}_2$ and $\mathfrak{S}_2 \times \mathfrak{S}_2$. It follows from Lemma 4.3.9 and Figure 1 on page 96, that the inverse images in $\mathrm{N}_{\mathbf{M}_c}(\mathbf{R}_{2,2})$ of these classes are represented by $\mathbf{R}_{2,2}$, $\mathbf{R}_{2,1,1}$, $\mathbf{R}_{1,1,2}$ and $\mathbf{R}_{1,1,1,1}$ in some order. $\square$

The previous results allow us to prove the following.

LEMMA 4.4.7. *Any radical* 2*-chain of* $\mathbf{M_c}$ *is* $\mathbf{M_c}$*-conjugate to a unique chain of the form*

$$C\colon P_0 = 1 < P_1 < \cdots < P_n, \tag{4.4.8}$$

where P_i*, for* $i = 0, 1, \ldots, n$*, is one of the radical* 2*-subgroups of* $\mathbf{M_c}$*, as given in Table 4.2 on page 107. Any chain of this form is a radical* 2*-chain of* $\mathbf{M_c}$*.*

PROOF. This is clear from Lemmas 4.4.1, 4.4.2, 4.4.5 and 4.4.6, Proposition 4.4.3, and the fact that $\mathbf{R}_{2,1,1}$, $\mathbf{R}_{1,2,1}$ and $\mathbf{R}_{1,1,2}$ are maximal in the Sylow 2-subgroup $\mathbf{R}_{1,1,1,1}$ of $\mathbf{M_c}$. □

The following proposition halves the number of radical 2-chains we must consider.

PROPOSITION 4.4.9. *The collection of radical* 2*-chains* (4.4.8) *of* $\mathbf{M_c}$ *containing any one of the groups* $\mathbf{R}_{1,1,1,1}$*,* $\mathbf{R}_{2,1,1}$*,* $\mathbf{R}_{1,1,2}$ *or* $\mathbf{R}_{2,2}$ *contribute zero to the alternating sums of Conjectures 1.4.2, 1.4.4 and 1.4.6.*

PROOF. Let C be a radical 2-chain of $\mathbf{M_c}$, as in (4.4.8). Then $\mathbf{T} \leq P_1$ since each non-trivial group in Table 4.2 contains $\mathbf{T}$.

If $\mathbf{T} < P_1$, form the 2-chain $C'\colon P_0 = 1 < \mathbf{T} < P_1 < P_2 < \cdots < P_n$ by inserting $\mathbf{T}$ between P_0 and P_1. If $\mathbf{T} = P_1$, form the 2-chain $C'\colon P_0 = 1 < P_2 < \cdots < P_n$ by deleting P_1 (if $n = 1$, we just drop P_n). In either case C' is a radical 2-chain of $\mathbf{M_c}$, since it has the same form as (4.4.8). Moreover the chains C and C' have opposite parity i.e. $|C| = |C'| \pm 1$.

Clearly if we apply the operation $'$ twice, we recover the original chain. In other words $(C')' = C$.

If $\mathrm{N}_{\mathbf{M_c}}(C) = \mathrm{N}_{\mathbf{M_c}}(C')$, then the contribution of C and C' to any alternating sum is zero. In particular this occurs if, for some value of $i = 1, 2, \ldots, n$ we have $\mathbf{T} < P_i$ and $\mathrm{N}_{\mathbf{M_c}}(P_i) \leq \mathrm{N}_{\mathbf{M_c}}(\mathbf{T}) = \mathbf{H}$. By Corollary 4.3.3 the normalizer of any one of the groups $\mathbf{R}_{1,1,1,1}$, $\mathbf{R}_{2,1,1}$ or $\mathbf{R}_{1,1,2}$ is contained in $\mathbf{H}$. The same is true of $\mathbf{R}_{2,2}$ by Lemma 4.3.9. Hence we can ignore all 2-chains containing any of these groups. □

Chain C	Chain Description	$\mathrm{N}_{\mathbf{M_c}}(C)$	$\mathrm{N}_{\mathbf{M_c}.2}(C)$	Parity
C_1	$\{1\}$	$\mathbf{M_c}$	$\mathbf{M_c}.2$	+
C_2	$\{1\} < \mathbf{R}_{3,1}$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$	-
C_3	$\{1\} < \mathbf{R}_{3,1} < \mathbf{R}_{1,2,1}$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$	+
C_4	$\{1\} < \mathbf{R}_{1,3}$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3})$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3})$	-
C_5	$\{1\} < \mathbf{R}_{1,3} < \mathbf{R}_{1,2,1}$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$	+
C_6	$\{1\} < \mathbf{R}_{1,2,1}$	$\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$	$\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})$	-
C_7	$\{1\} < \mathbf{T}$	$\mathbf{H}$	$\mathbf{H}.2$	-
C_8	$\{1\} < \mathbf{T} < \mathbf{R}_{3,1}$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$	+
C_9	$\{1\} < \mathbf{T} < \mathbf{R}_{3,1} < \mathbf{R}_{1,2,1}$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$	-
C_{10}	$\{1\} < \mathbf{T} < \mathbf{R}_{1,3}$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,3})$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,3})$	+
C_{11}	$\{1\} < \mathbf{T} < \mathbf{R}_{1,3} < \mathbf{R}_{1,2,1}$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$	-
C_{12}	$\{1\} < \mathbf{T} < \mathbf{R}_{1,2,1}$	$\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$	$\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1})$	+

TABLE 4.3. The Remaining Radical 2-chains of $\mathbf{M_c}$

We can now give a list of radical 2-chains of $\mathbf{M_c}$ which is sufficient for the purpose of proving the Conjectures.

THEOREM 4.4.10. *If C is a radical 2-chain of $\mathbf{M_c}$ then C is redundant from the point of view of the Conjectures, or C is $\mathbf{M_c}$-conjugate to exactly one of the radical 2-chains given in Table 4.3 on the preceding page. The only chains in this table which are conjugate in $\mathbf{M_c}.2$ are the pairs $\{C_2, C_4\}$, $\{C_3, C_5\}$, $\{C_8, C_{10}\}$ and $\{C_9, C_{11}\}$.*

PROOF. This is clear from Section 4.3, Lemma 4.4.7 and Proposition 4.4.9. $\square$

Note that the 2-chains $C_1, \ldots, C_6$ in Table 4.3 are not the same as the 3-chains having the same names in Table 3.1 on page 48.

4.5. The Character Degrees of H

From the character table of $\mathbf{H} \cong 2.\mathfrak{A}_8$ given in [**Con85**, p22], the group $\mathrm{N}_{\mathbf{M_c}}(C_7) = \mathbf{H}$ has two 2-blocks. The characters χ_{13} and χ_{23} lie in a block b_1 of defect 1, while the remaining 21 characters lie in the principal 2-block b_0. Clearly $\mathbf{T}$ is the unique defect group of b_1. Since $\mathbf{H} = \mathrm{N}_{\mathbf{M_c}}(\mathbf{T})$, we conclude from Brauer's First Main Theorem that ${\mathsf{b}_1}^{\mathbf{M_c}}$ is a block of $\mathbf{M_c}$ of defect 1. The characters of b_1 are handled by the theory of blocks with cyclic defect groups.

We list here the characters in b_0 and their defects:

Character	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}
Degree	1	7	14	20	21	21	21	28	35	45	45
Defect	7	7	6	5	7	7	7	5	7	7	7
Character	χ_{12}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	
Degree	56	70	8	24	24	48	56	56	56	56	
Defect	4	6	4	4	4	3	4	4	4	4	

Since $\mathrm{N}_{\mathbf{M_c}}(C_7) = \mathbf{H}$, we have

$$\begin{aligned} &\mathrm{k}(C_7, \mathsf{B}_0, 7) = 8, \qquad \mathrm{k}(C_7, \mathsf{B}_0, 6) = 2, \\ &\mathrm{k}(C_7, \mathsf{B}_0, 5) = 2, \qquad \mathrm{k}(C_7, \mathsf{B}_0, 4) = 8, \\ &\mathrm{k}(C_7, \mathsf{B}_0, 3) = 1, \qquad \mathrm{k}(C_7, \mathsf{B}_0, d) = 0, \quad \text{for all other values of } d. \end{aligned} \tag{4.5.1}$$

From [**Con85**, p22] the characters of b_0 invariant in $\mathbf{H}.2$ are χ_1, χ_2, χ_3, χ_4, χ_5, χ_8, χ_9, χ_{12}, χ_{14}, χ_{15} and χ_{18}.

Thus

$$\begin{aligned} &\mathrm{k}(C_7, \mathsf{B}_0, 7, \overline{\mathbf{M_c}.2}) = 4, \quad \mathrm{k}(C_7, \mathsf{B}_0, 6, \overline{\mathbf{M_c}.2}) = 2, \\ &\mathrm{k}(C_7, \mathsf{B}_0, 5, \overline{\mathbf{M_c}.2}) = 2, \quad \mathrm{k}(C_7, \mathsf{B}_0, 4, \overline{\mathbf{M_c}.2}) = 2, \\ &\mathrm{k}(C_7, \mathsf{B}_0, 3, \overline{\mathbf{M_c}.2}) = 1, \quad \mathrm{k}(C_7, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}) = 0, \quad \text{for all other values of } d. \end{aligned} \tag{4.5.2}$$

4.6. The Character Degrees of $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$ and $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$

We note that $\mathbf{R}_{3,1}$ is a self-centralizing 2-subgroup of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$. Hence the principal 2-block is the unique 2-block of each of the groups $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$ and $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$.

So the characters of both groups lie in 2-blocks inducing the principal 2-block, B_0, of $\mathbf{M_c}$.

Recall from Lemma 4.1.5 and Table 4.1 on page 97 that $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}) = \mathbf{R}_{3,1} \rtimes \mathrm{G}_{3,1}$, where $\mathrm{G}_{3,1} \cong \mathrm{GL}(3,2)$ acts as the full automorphsim group on $\mathrm{A}_{3,1} = \mathbf{R}_{3,1}/\mathbf{T}$.

Lemma 4.6.1. $\mathrm{Orb}(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}), \mathbf{R}^*_{3,1}) = \{1, 7, 8\}$.

Proof. The group $\mathrm{G}_{3,1}$ acts transitively on the subspaces of codimension 1 of $\mathrm{A}_{3,1}$. Therefore the 7 elements of $\mathrm{Irr}(\mathbf{R}_{3,1} \bmod \mathbf{T})^{\#}$ form a single $\mathrm{G}_{3,1}$-orbit. The trivial character forms another $\mathrm{G}_{3,1}$-orbit on $\mathrm{Irr}(\mathbf{R}_{3,1} \bmod \mathbf{T})$.

Since $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$ acts transitively on $\mathbf{R}_{3,1}\backslash\mathbf{T}$ (by Corollary 4.1.6) and fixes both elements of $\mathbf{T}$, there are exactly three $\mathrm{G}_{3,1}$-orbits in $\mathbf{R}_{3,1}$. So there are just three $\mathrm{G}_{3,1}$-orbits in $\mathrm{Irr}(\mathbf{R}_{3,1})$. We have just seen that $\mathrm{Irr}(\mathbf{R}_{3,1} \bmod \mathbf{T})$ accounts for two of these orbits. So the complement $\mathrm{Irr}(\mathbf{R}_{3,1} \mid \mathbf{T})$ to $\mathrm{Irr}(\mathbf{R}_{3,1} \bmod \mathbf{T})$ is the third orbit. □

We now deal with the chains C_8 and C_{10}.

Proposition 4.6.2. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_8) = \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$ *has a unique* 2*-block, which necessarily induces the principal* 2*-block,* B_0*, of* $\mathbf{M_c}$*. Moreover*

$$\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})\big) = \{1, 3^2, 6, 7^3, 8^4, 14, 21^2, 24^2\} \text{ and } \mathrm{Def}_2\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})\big) = \{7^8, 6^2, 4^6\}.$$

Thus

$$\begin{aligned} &\mathrm{k}(C_8, \mathsf{B}_0, 7) = 8, && \mathrm{k}(C_8, \mathsf{B}_0, 6) = 2, \\ &\mathrm{k}(C_8, \mathsf{B}_0, 5) = 0, && \mathrm{k}(C_8, \mathsf{B}_0, 4) = 6, \\ &\mathrm{k}(C_8, \mathsf{B}_0, 3) = 0, && \mathrm{k}(C_8, \mathsf{B}_0, d) = 0, \quad \textit{for all other values of } d. \end{aligned} \tag{4.6.3}$$

Also

$$\text{(4.6.4)} \qquad \mathrm{k}(C_{10}, \mathsf{B}_0, d) = \mathrm{k}(C_8, \mathsf{B}_0, d), \quad \textit{for all values of } d.$$

Proof. By Lemma 4.6.1 the group $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$ has two orbits on $\mathrm{Irr}(\mathbf{R}_{3,1})^{\#}$, one of length 7 and the other of length 8. We let ψ_1 be a representative of the former orbit, and ψ_2 be a representative of the latter orbit.

From the Atlas, the group $\mathrm{G}_{3,1} \cong \mathrm{GL}(3,2)$ has two conjugacy classes of maximal subgroups with index 7. Each consists of subgroups isomorphic to $\mathfrak{S}_4$. The stabilizer of ψ_1 in $\mathrm{G}_{3,1}$ must lie in one of these classes of subgroups. By Theorem 1.2.15, the character ψ_1 extends to its stabilizer in $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$. It then follows from Theorem 1.2.16 that $\mathrm{Deg}\big(\mathrm{I}_{\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})}(\psi_1) \mid \psi_1\big) = \mathrm{Deg}(\mathfrak{S}_4) = \{1^2, 2, 3^2\}$. Using Clifford Theory we conclude

$$\text{(4.6.5)} \qquad \mathrm{Deg}\big(\ \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}) \mid \psi_1\ \big) = \{7^2, 14, 21^2\}.$$

From the Atlas, the group $\mathrm{G}_{3,1} \cong \mathrm{GL}(3,2)$ has a unique conjugacy class of subgroups with index 8, consisting of subgroups isomorphic to $7\colon 3$. Moreover the group $7\colon 3$ is Frobenius (for instance because $\mathrm{GL}(3,2)$ has no elements of order 21). The stabilizer of ψ_2 in $\mathrm{G}_{3,1}$ must lie in this class of subgroups. By Theorem 1.2.15, the character ψ_2 extends to its stabilizer in $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$. It then follows from Theorem 1.2.16 that $\mathrm{Deg}\big(\mathrm{I}_{\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})}(\psi_2) \mid \psi_2\big) = \mathrm{Deg}(7\colon 3) = \{1^3, 3^2\}$. Using Clifford Theory we conclude

$$\text{(4.6.6)} \qquad \mathrm{Deg}\big(\ \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}) \mid \psi_2\ \big) = \{8^3, 24^2\}.$$

Since $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1}) \cong \mathbf{R}_{3,1} \rtimes \mathrm{GL}(3,2)$, we also have

$$\text{(4.6.7)} \qquad \mathrm{Deg}\big(\ \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})/\mathbf{R}_{3,1}\ \big) = \mathrm{Deg}\big(\ \mathrm{GL}(3,2)\ \big) = \{1, 3^2, 6, 7, 8\}.$$

Equations (4.6.3) now follow from (4.6.5), (4.6.6) and (4.6.7). Equation (4.6.4) holds because the 2-chains C_8 and C_{10} are conjugate in $\mathbf{M_c}.2$. □

We now consider the chains C_2 and C_4.

LEMMA 4.6.8. $\mathrm{Orb}(\Lambda_{3,1}, \mathbf{R}^*_{3,1}) = \{1, 15\}$. *So the stabilizer in* $\Lambda_{3,1}$ *of any non-trivial character of* $\mathbf{R}_{3,1}$ *is isomorphic to* $\mathrm{GL}(3,2)$. *Hence*

$$\text{(4.6.9)} \qquad \mathrm{Deg}\big(\ \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) \mid \mathbf{R}_{3,1}\ \big) = \{15, 45^2, 90, 105, 120\}.$$

PROOF. By Corollary 4.3.11, the group $\Lambda_{3,1}$ acts transitively on the subspaces of $\mathbf{R}_{3,1}$ of codimension 1. Hence it also acts transitively on $(\mathbf{R}^*_{3,1})^{\#}$. From the Atlas, there are two conjugacy classes of subgroups of index $|(\mathbf{R}^*_{3,1})^{\#}| = 15$ in $\Lambda_{3,1} \cong \mathfrak{A}_7$. A representative of either of these classes is isomorphic to $\mathrm{GL}(3,2)$. Hence the stabilizer in $\Lambda_{3,1}$ of any non-trivial character of $\mathbf{R}_{3,1}$ is isomorphic to $\mathrm{GL}(3,2)$. Clifford Theory now gives us (4.6.9). □

The following completes our analysis of the stabilizers of the radical 2-chains C_2 and C_4.

PROPOSITION 4.6.10. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_2) = \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$ *has a unique* 2*-block, which necessarily induces the principal* 2*-block,* B_0*, of* $\mathbf{M_c}$. *Moreover* $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})\big) =$

$\{1,6,10^2,14^2,15^2,21,35,45^2,90,105,120\}$ *and* $\mathrm{Def}_2\big(\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})\big) = \{7^8, 6^6, 4\}$. *Hence*

$$\begin{aligned} &\mathrm{k}(C_2, \mathsf{B}_0, 7) = 8, &&\mathrm{k}(C_2, \mathsf{B}_0, 6) = 6, \\ &\mathrm{k}(C_2, \mathsf{B}_0, 5) = 0, &&\mathrm{k}(C_2, \mathsf{B}_0, 4) = 1, \\ &\mathrm{k}(C_2, \mathsf{B}_0, 3) = 0, &&\mathrm{k}(C_2, \mathsf{B}_0, d) = 0, \quad \textit{for all other values of } d. \end{aligned} \tag{4.6.11}$$

Also

$$\mathrm{k}(C_4, \mathsf{B}_0, d) = \mathrm{k}(C_2, \mathsf{B}_0, d), \quad \textit{for all values of } d. \tag{4.6.12}$$

PROOF. Since $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})/\mathbf{R}_{3,1} \cong \mathfrak{A}_7$, we have

$$\mathrm{Deg}\big(\ \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})/\mathbf{R}_{3,1}\ \big) = \mathrm{Deg}(\mathfrak{A}_7) = \{1, 6, 10^2, 14^2, 15, 21, 35\}. \tag{4.6.13}$$

Equations (4.6.11) now follow from (4.6.9) and (4.6.13). Equation (4.6.12) holds because the 2-chains C_2 and C_4 are conjugate in $\mathbf{M_c}.2$. $\square$

4.7. The Character Degrees of $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ and $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$

It follows from Lemma 4.3.14 that $\mathrm{C}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) = \mathrm{C}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) = \mathrm{Z}(\mathbf{R}_{1,2,1})$ is contained in $\mathbf{R}_{1,2,1}$. Hence the principal 2-block is the unique 2-block of each of the groups $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ and $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$. So the characters of both groups lie in 2-blocks inducing the principal 2-block, B_0, of $\mathbf{M_c}$.

We deal first with the stabilizer $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ of the radical 2-chains C_9, C_{11} and C_{12}.

LEMMA 4.7.1. $\mathbf{R}_{1,2,1}$ *has exactly three irreducible characters which are non-trivial on* $\mathrm{Z}(\mathbf{R}_{1,2,1})$. *Let* χ *be one such. Then* χ *vanishes outside* $\mathrm{Z}(\mathbf{R}_{1,2,1})$ *and* $\chi|_{\mathrm{Z}(\mathbf{R}_{1,2,1})} =$

4λ, where λ is a non-trivial linear character of $\mathrm{Z}(\mathbf{R}_{1,2,1})$. The group $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ acts doubly transitively on the three elements of $\mathrm{Irr}\big(\mathbf{R}_{1,2,1} \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big)$, while the normalizer $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ of $\mathbf{T} \leq \mathrm{Z}(\mathbf{R}_{1,2,1})$ in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ acts transitively on the two elements of $\mathrm{Irr}\big(\mathbf{R}_{1,2,1} \mid \mathbf{T}\big) \subseteq \mathrm{Irr}\big(\mathbf{R}_{1,2,1} \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big)$.

PROOF. Let λ be a non-trivial character of $\mathrm{Z}(\mathbf{R}_{1,2,1})$. By Corollary 4.3.15 the group $\mathbf{R}_{1,2,1}/\operatorname{Ker}(\lambda)$ is extra-special of type 2^{1+4}. It follows that $\mathrm{Irr}(\mathbf{R}_{1,2,1} \mid \lambda)$ consists of a single irreducible character χ, which vanishes outside $\mathrm{Z}(\mathbf{R}_{1,2,1})$, and equals 4λ on $\mathrm{Z}(\mathbf{R}_{1,2,1})$.

Since there are three choices for λ, the group $\mathbf{R}_{1,2,1}$ has exactly three irreducible characters lying over non-trivial characters of its center.

By Lemma 4.3.14 the group $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ acts doubly transitively on $\mathrm{Z}(\mathbf{R}_{1,2,1})^{\#}$. It follows that it acts doubly transitively on $\mathrm{Irr}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big)^{\#}$, and that the normalizer $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ of $\mathbf{T}$ acts transitively on the subset $\mathrm{Irr}\big(\mathrm{Z}(\mathbf{R}_{1,2,1}) \mid \mathbf{T}\big)$ of $\mathrm{Irr}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big)^{\#}$. This implies the rest of the lemma. $\square$

We now compute the degrees of the irreducible characters of $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ lying over the single non-trivial character of $\mathrm{Z}(\mathbf{R}_{1,2,1})$ containing $\mathbf{T}$ in its kernel.

LEMMA 4.7.2. $\mathrm{Deg}\big(\,\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})/\mathbf{T}\,\big) = \{4^2, 8\}$.

PROOF. From Lemma 4.7.1, it is clear that $\mathbf{R}_{1,2,1}$ has a unique irreducible character χ_1 lying over the unique non-trivial linear character in $\mathrm{Irr}(\mathrm{Z}(\mathbf{R}_{1,2,1}) \bmod \mathbf{T})$. Also χ_1 is invariant in the group $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$. Then χ_1 extends to this group, since all Sylow subgroups of $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})/\mathbf{R}_{1,2,1} \cong \mathfrak{S}_3$ are cyclic. The lemma now follows from Clifford Theory. $\square$

Next we compute the degrees of all irreducible characters of $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ lying over non-trivial characters of $\mathbf{R}_{3,1}$.

LEMMA 4.7.3.

$$\mathrm{Deg}\big(\ \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) \mid \mathbf{R}_{3,1}\ \big) = \{3^4, 4^2, 6, 8^4\},\ \textit{and}$$

$$\mathrm{Deg}\big(\ \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) \mid \mathbf{R}_{3,1}/\mathbf{T}\ \big) = \{3^4, 4^2, 6, 8\}.$$

PROOF. By Lemma 4.7.1 the two elements of $\mathrm{Irr}(\ \mathrm{Z}(\mathbf{R}_{1,2,1}) \mid \mathbf{T}\ )$ form a single $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$-orbit. Let χ_2 be a character in this orbit. Then $\mathrm{I}_{\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})}(\chi_2) \cong \mathbf{R}_{1,2,1}.\mathbb{Z}_3$. From Clifford Theory we get

$$\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) \mid \mathbf{T}\big) = \{8^3\}. \tag{4.7.4}$$

The group $\mathrm{N}_{1,2,1}$ acts flag transitively on $\mathrm{A}_{3,1}/\langle\varepsilon\rangle$, by (4.2.1). Hence there is a single $\mathrm{N}_{1,2,1}$-orbit of non-trivial linear characters of $\mathrm{A}_{3,1}$ whose kernels contain $\langle\varepsilon\rangle$. Let χ_3 be a member of this orbit. Identify χ_3 with its inflation to $\mathbf{R}_{3,1}$. Then $\mathrm{I}_{\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})}(\chi_3)$ has index 3 in $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$. So $\mathrm{I}_{\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})}(\chi_3) \cong \mathbf{R}_{3,1} \rtimes D_8$. Since $\mathbf{R}_{3,1}$ is abelian, it follows immediately from Clifford Theory that

$$\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) \mid \mathbf{R}_{3,1}/\,\mathrm{Z}(\mathbf{R}_{1,2,1})\big) = \{3^4, 6\}. \tag{4.7.5}$$

Now the set $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) \mid \mathbf{R}_{3,1}\big)$ is obtained from Lemma 4.7.2 and Equations (4.7.4) and (4.7.5), while $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) \mid \mathbf{R}_{3,1}/\mathbf{T}\big)$ is obtained from Lemma 4.7.2 and Equation (4.7.5). $\square$

We can now prove

PROPOSITION 4.7.6. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_9) = \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ *has a unique* 2*-block, which necessarily induces the principal* 2*-block,* B_0*, of* $\mathbf{M_c}$*. Moreover*

$$\begin{aligned} \mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})/\mathbf{T}\big) &= \{1^2, 2, 3^6, 4^2, 6, 8\}, \\ \mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})\big) &= \{1^2, 2, 3^6, 4^2, 6, 8^4\}, \qquad (4.7.7) \\ \mathrm{Def}_2\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})\big) &= \{7^8, 6^2, 5^2, 4^4\}. \end{aligned}$$

Hence

$$\begin{aligned} &\mathrm{k}(C_9, \mathsf{B}_0, 7) = 8, \qquad \mathrm{k}(C_9, \mathsf{B}_0, 6) = 2, \\ &\mathrm{k}(C_9, \mathsf{B}_0, 5) = 2, \qquad \mathrm{k}(C_9, \mathsf{B}_0, 4) = 4, \qquad (4.7.8) \\ &\mathrm{k}(C_9, \mathsf{B}_0, 3) = 0, \qquad \mathrm{k}(C_9, \mathsf{B}_0, d) = 0, \quad \textit{for all other values of } d. \end{aligned}$$

Also

$$\mathrm{k}(C_9, \mathsf{B}_0, d) = \mathrm{k}(C_{11}, \mathsf{B}_0, d) = \mathrm{k}(C_{12}, \mathsf{B}_0, d), \quad \textit{for all values of } d. \qquad (4.7.9)$$

PROOF. Lemma 4.7.3 dealt with all characters of $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ non-trivial on $\mathbf{R}_{3,1}$. Since $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) = \mathbf{R}_{3,1} \rtimes \mathfrak{S}_4$, the remaining characters are just the characters of $\mathfrak{S}_4$ inflated to $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$. Equations (4.7.7) follow from this and Lemma 4.7.3. Since $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ has a unique 2-block, we immediately obtain (4.7.8). The equalities in (4.7.9) follow from the fact that $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ is the stabilizer of each of the radical 2-chains C_9, C_{11} and C_{12} in $\mathbf{M_c}$. $\square$

We now consider the stabilizer $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})\big)$ of the radical 2-chains C_3, C_5 and C_6.

LEMMA 4.7.10. $\mathrm{Deg}\big(\, \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\,\big) = \{12^2, 24\}$.

PROOF. By Lemma 4.7.1, the group $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ acts transitively on the three elements of $\mathrm{Irr}\big(\mathbf{R}_{1,2,1} \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big)$. Since an element of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ fixes a linear character λ of $\mathrm{Z}(\mathbf{R}_{1,2,1})$ if and only if it normalizes $\mathrm{Ker}(\lambda)$, it follows from Lemma 4.7.1 that the unique character $\chi \in \mathrm{Irr}\big(\mathbf{R}_{1,2,1} \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big)$ with kernel $\mathbf{T}$ has $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ as its stabilizer in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$. The result now follows from Lemma 4.7.2 and Clifford Theory. $\square$

COROLLARY 4.7.11. *Let* $\chi \in \mathrm{Irr}\big(\mathbf{R}_{1,2,1} \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big)$. *Then the stabilizer* $\mathrm{I}_{\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})}(\chi)$ *of* χ *in* $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ *is conjugate to* $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$. *Hence the stabilizer has Sylow* 3*-subgroups of order* 3.

PROOF. The proof of the lemma shows that $\mathrm{I}_{\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})}(\chi)$ is conjugate to $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$. This last group has Sylow 3-subgroups of order 3, since $\mathbf{R}_{1,2,1}$ is a 2-group, and the quotient group $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})/\mathbf{R}_{1,2,1} \cong \mathrm{N}_{1,2,1}/\mathrm{A}_{1,2,1}$ is isomorphic to $\mathfrak{S}_3$ by (4.2.1). $\square$

LEMMA 4.7.12. $\mathrm{Deg}\big(\, \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})/\, \mathrm{Z}(\mathbf{R}_{1,2,1})\,\big) = \{1^2, 2^4, 3^4, 6^2, 9^2\}$.

PROOF. We give detailed descriptions of the structure of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1})$ in the beginning of Section 4.9. It turns out to be expedient to quote these results here, although we could prove the present lemma without this material.

It follows from Corollary 4.9.5 that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1}) = \big(\mathrm{A_4}^{(1)} \times \mathrm{A_4}^{(2)}\big) \rtimes \langle \iota \rangle$, where $\mathrm{A_4}^{(i)} \cong \mathfrak{A}_4$ and $\mathrm{A_4}^{(i)} \rtimes \langle \iota \rangle \cong \mathfrak{S}_4$ for $i = 1, 2$. The result of the lemma is now a straightforward exercise in Clifford Theory. $\square$

The next result will be needed in Section 4.10.

Corollary 4.7.13. *Let* $\chi \in \operatorname{Irr}(\mathbf{R}_{1,2,1})^{\#}$. *Then the stabilizer* $\mathrm{I}_{\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})}(\chi)$ *has Sylow* 3*-subgroups of order* 1 *or* 3.

Proof. If χ is an element of $\operatorname{Irr}\big(\mathbf{R}_{1,2,1} \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big)$, then this is a consequence of Corollary 4.7.11.

If χ is trivial on $\mathrm{Z}(\mathbf{R}_{1,2,1})$, then it is inflated from a non-trivial character $\overline{\chi}$ of $\mathbf{R}_{1,2,1}/\,\mathrm{Z}(\mathbf{R}_{1,2,1})$. In the notation of the proof of the lemma, this last group is the direct product $\mathrm{V}_1 \times \mathrm{V}_2$ of the four-groups $\mathrm{V}_1 = \mathrm{O}_2(\mathrm{A_4}^{(1)})$ and $\mathrm{V}_2 = \mathrm{O}_2(\mathrm{A_4}^{(2)})$. So $\overline{\chi}$ is the direct product $\overline{\chi}_1 \times \overline{\chi}_2$ of some linear characters $\overline{\chi}_1$ and $\overline{\chi}_2$ of V_1 and V_2, respectively. A Sylow 3-subgroup of its stabilizer in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1}) = \big(\mathrm{A_4}^{(1)} \times \mathrm{A_4}^{(2)}\big) \rtimes \langle \iota \rangle$ has order strictly bigger than 3 if and only if $\mathrm{A_4}^{(i)}$ stabilizes $\overline{\chi}_i$ for $i = 1, 2$. This happens if and only if $\overline{\chi}_1$ and $\overline{\chi}_2$ are both trivial, in which case $\overline{\chi}$ is trivial, contradicting our hypothesis. $\square$

We can now prove the following

Proposition 4.7.14. *The group* $\mathrm{N}_{\mathbf{M_c}}(C_3) = \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ *has a unique* 2*-block, which necessarily induces the principal* 2*-block,* B_0*, of* $\mathbf{M_c}$*. Moreover*

$$\operatorname{Deg}\big(\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})\big) = \{1^2, 2^4, 3^4, 6^2, 9^2, 12^2, 24\}, \quad \textit{and}$$

$$\operatorname{Def}_2\big(\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})\big) = \{7^8, 6^6, 5^2, 4\}.$$

Hence

$$\text{(4.7.15)} \qquad \begin{aligned} &\mathrm{k}(C_3, \mathsf{B}_0, 7) = 8, && \mathrm{k}(C_3, \mathsf{B}_0, 6) = 6, \\ &\mathrm{k}(C_3, \mathsf{B}_0, 5) = 2, && \mathrm{k}(C_3, \mathsf{B}_0, 4) = 1, \\ &\mathrm{k}(C_3, \mathsf{B}_0, 3) = 0, && \mathrm{k}(C_3, \mathsf{B}_0, d) = 0, \quad \textit{for all other values of } d. \end{aligned}$$

Also

$$\text{(4.7.16)} \qquad \mathrm{k}(C_3, \mathsf{B}_0, 7) = \mathrm{k}(C_5, \mathsf{B}_0, 7) = \mathrm{k}(C_6, \mathsf{B}_0, 7), \quad \textit{for all values of } d.$$

Proof. We obtain the degrees of the irreducible characters of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ from Lemmas 4.7.10 and 4.7.12. Then (4.7.15) is an immediate consequence, given the fact that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ has a unique 2-block.

The equalities in (4.7.16) follow from the fact that the radical 2-chains C_3, C_5 and C_6 have identical stabilizers in $\mathbf{M_c}$. □

4.8. The Ordinary Conjecture for the prime $p = 2$

From [**Con85**, p101] the group $\mathbf{M_c}$ has four 2-blocks of defect 0, one block of defect 1 containing two ordinary characters, and the principal block B_0 containing the remaining 18 characters. The conjecture holds for the block of defect 1 by the theory of blocks with cyclic defect. For details see [**Da96**]. So we need only worry about the principal block. We list here the characters of the principal block and their defects, with the notation taken from [**Con85**]:

Character	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_9	χ_{12}	χ_{13}
Degree	1	22	231	252	770	770	1750	4500	4752
Defect	7	6	7	5	6	6	6	5	3
Character	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{23}	χ_{24}
Degree	5103	5544	8019	8019	8250	8250	9625	10395	10395
Defect	7	4	7	7	6	6	7	7	7

Thus, since $\mathrm{N}_{\mathbf{M_c}}(C_1) = \mathbf{M_c}$, we have

$$\begin{aligned} &\mathrm{k}(C_1, \mathsf{B}_0, 7) = 8, \qquad \mathrm{k}(C_1, \mathsf{B}_0, 6) = 6, \\ &\mathrm{k}(C_1, \mathsf{B}_0, 5) = 2, \qquad \mathrm{k}(C_1, \mathsf{B}_0, 4) = 1, \\ &\mathrm{k}(C_1, \mathsf{B}_0, 3) = 1, \qquad \mathrm{k}(C_1, \mathsf{B}_0, d) = 0, \quad \text{for all other values of } d. \end{aligned} \tag{4.8.1}$$

From [**Con85**, p101] the characters χ_1, χ_2, χ_3, χ_4, χ_9, χ_{12}, χ_{13}, χ_{14}, χ_{15}, χ_{20} are the characters of B_0 invariant in $\mathbf{M_c}.2$. Thus

$$\begin{aligned} &\mathrm{k}(C_1, \mathsf{B}_0, 7, \overline{\mathbf{M_c}.2}) = 4, \quad \mathrm{k}(C_1, \mathsf{B}_0, 6, \overline{\mathbf{M_c}.2}) = 2, \\ &\mathrm{k}(C_1, \mathsf{B}_0, 5, \overline{\mathbf{M_c}.2}) = 2, \quad \mathrm{k}(C_1, \mathsf{B}_0, 4, \overline{\mathbf{M_c}.2}) = 1, \\ &\mathrm{k}(C_1, \mathsf{B}_0, 3, \overline{\mathbf{M_c}.2}) = 1, \quad \mathrm{k}(C_1, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}) = 0, \quad \text{for all other values of } d. \end{aligned} \tag{4.8.2}$$

We now have enough information for the following theorem.

THEOREM 4.8.3. *The Ordinary Conjecture holds for McLaughlin's simple group and the prime $p = 2$.*

PROOF. From Conjecture 1.4.2 and Table 4.3 on page 110 we need to prove

$$\begin{aligned}&\mathrm{k}(C_1, \mathsf{B}_0, d) + \mathrm{k}(C_3, \mathsf{B}_0, d) + \mathrm{k}(C_8, \mathsf{B}_0, d) + \mathrm{k}(C_{10}, \mathsf{B}_0, d) = \\ &\mathrm{k}(C_2, \mathsf{B}_0, d) + \mathrm{k}(C_4, \mathsf{B}_0, d) + \mathrm{k}(C_7, \mathsf{B}_0, d) + \mathrm{k}(C_9, \mathsf{B}_0, d),\end{aligned} \tag{4.8.4}$$

for all values of $d \in \mathbb{Z}$.

From (4.8.1), (4.7.15), (4.6.3), (4.6.4), (4.6.11), (4.6.12), (4.5.1) and (4.7.8) we obtain the following sums for the above equation for various values of d:

2-Defect	C_1		C_3		C_8		C_{10}		C_2		C_4		C_7		C_9
7	8	+	8	+	8	+	8	=	8	+	8	+	8	+	8
6	6	+	6	+	2	+	2	=	6	+	6	+	2	+	2
5	2	+	2	+	0	+	0	=	0	+	0	+	2	+	2
4	1	+	1	+	6	+	6	=	1	+	1	+	8	+	4
3	1	+	0	+	0	+	0	=	0	+	0	+	1	+	0

TABLE 4.4. The Ordinary Conjecture for $p = 2$

The summands in Equation (4.8.4) are zero for all other values of d. This completes the proof. ☐

4.9. The Invariant conjecture for the prime $p = 2$

In view of Table 4.3, and Equations (4.8.2) and (4.5.2), it remains only to compute the degrees of the characters of $\mathrm{N}_{\mathbf{M_c}}(C_6) = \mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ and $\mathrm{N}_{\mathbf{M_c}}(C_{12}) = \mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$, which are invariant in $\mathrm{N}_{\mathbf{M_c}.2}(C_6) = \mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})$ and $\mathrm{N}_{\mathbf{M_c}.2}(C_{12}) = \mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1})$ respectively.

First we introduce some notation, which we will use for the remainder of this chapter.

Let $\mathrm{V} = \mathbf{R}_{1,2,1}/\,\mathrm{Z}(\mathbf{R}_{1,2,1})$, $\mathrm{V}_1 = \mathbf{R}_{3,1}/\,\mathrm{Z}(\mathbf{R}_{1,2,1})$ and $\mathrm{V}_2 = \mathbf{R}_{1,3}/\,\mathrm{Z}(\mathbf{R}_{1,2,1})$. So V is elementary abelian of order 2^4, and $\mathrm{V} = \mathrm{V}_1 \times \mathrm{V}_2$.

Recall from Section 3.1 that $\mathbf{E}$ is an elementary abelian 3-subgroup of $\mathbf{M_c}$ of order 3^4, and $\mathrm{N}_{\mathbf{M_c}}(\mathbf{E}) = \mathbf{E} \rtimes \mathbf{M}$, where $\mathbf{M} \cong \mathbf{M}_{10}$. By Proposition 3.1.12 there exists an element β_1 of $\mathbf{E}$ which comes from the $(3B)$ conjugacy class of $\mathbf{M_c}$, such that $\mathrm{C}_{\mathbf{M_c}}(\beta_1) = \mathbf{E} \rtimes \mathrm{A}_4$, where $\mathrm{A}_4 \cong \mathfrak{A}_4$ is contained in $\mathbf{M}$. Also $\mathrm{A}_4 < \mathrm{N}_{\mathbf{M}}\big(\langle\beta_1\rangle\big) = \mathrm{S}_4$, where $\mathrm{S}_4 \cong \mathfrak{S}_4$ is contained in $\mathbf{M}' \cong \mathfrak{A}_6$. Clearly each element of $\mathrm{S}_4\backslash\mathrm{A}_4$ inverts β_1.

Let β_2 be an element of order 3 in A_4, and let ι be an involution in $\mathrm{S}_4\backslash\mathrm{A}_4$ inverting β_2. Then ι inverts β_1. Furthermore S_4 is the semi-direct product of the four-group $\mathrm{K} = \mathrm{O}_2(\mathrm{A}_4)$ with $\langle\beta_2, \iota\rangle \cong \mathfrak{S}_3$.

Let $\mathrm{S} = \langle\beta_1, \beta_2\rangle$. Then S is an elementary abelian 3-subgroup of $\mathbf{E} \rtimes \mathbf{M}$ of order 3^2 inverted by ι. Also $\mathrm{S} \rtimes \langle\iota\rangle$ normalizes K, but intersects it trivially. Hence $\mathbf{E} \rtimes \mathbf{M}$ contains the group $\mathrm{K} \rtimes \big(\mathrm{S} \rtimes \langle\iota\rangle\big)$.

Now $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{E}) = \mathbf{E} \rtimes \big(\mathbf{M} \times \langle c\rangle\big)$, where c is an involution in $\mathbf{M_c}.2\backslash\mathbf{M_c}$ inverting $\mathbf{E}$, by Lemma 3.7.1. Then c inverts $\beta_1 \in \mathbf{E}$ and centralizes $\mathrm{S}_4 = \mathrm{K} \rtimes \langle\beta_2, \iota\rangle \leq \mathbf{M}$. It follows that $\mathrm{K}^{(1)} = \langle\iota, c\rangle$ is a four-subgroup of $\mathbf{M_c}.2$ normalizing $\langle\beta_1\rangle$, $\langle\beta_2\rangle$ and K. So we can form the subgroup $\mathrm{K} \rtimes \big(\mathrm{S} \rtimes \mathrm{K}^{(1)}\big)$ of $\mathbf{E} \rtimes \big(\mathbf{M} \times \langle c\rangle\big)$.

By Proposition 4.1.3, the four-groups K and $\mathrm{Z}(\mathbf{R}_{1,2,1})$ are conjugate in $\mathbf{M_c}$. Conjugating by an element of $\mathbf{M_c}$, if necessary, we may assume that $\mathrm{K} = \mathrm{Z}(\mathbf{R}_{1,2,1})$. The same proposition also shows that $\mathrm{N}_{\mathbf{M_c}}(\mathrm{K})$ acts transitively on $\mathrm{K}^{\#}$. Conjugating by an element of $N_{\mathbf{M_c}}(K)$, if necessary, we may assume that $\tau \in Z(\mathbf{R}_{1,2,1}) = K$, is centralized by $\iota \in K^{(1)}$.

We summarise our notation in the following table:

$V \cong \mathbb{Z}_2^4$	The elementary abelian group $\mathbf{R}_{1,2,1}/Z(\mathbf{R}_{1,2,1})$.
$V_1 \cong \mathbb{Z}_2^2$	The subgroup $\mathbf{R}_{3,1}/Z(\mathbf{R}_{1,2,1})$ of V.
$V_2 \cong \mathbb{Z}_2^2$	The subgroup $\mathbf{R}_{1,3}/Z(\mathbf{R}_{1,2,1})$ of V.
$\mathbf{E} \cong \mathbb{Z}_3^4$	A subgroup of $\mathbf{M_c}$ isomorphic to $\mathbb{Z}_3^4$.
$\mathbf{M} \cong \mathbf{M}_{10}$	A complement to $\mathbf{E}$ in $N_{\mathbf{M_c}}(\mathbf{E})$.
β_1	A fixed element of $\mathbf{E}$ from the $(3B)$ class of $\mathbf{M_c}$.
$A_4 \cong \mathfrak{A}_4$	The centralizer of β_1 in $\mathbf{M}$.
$S_4 \cong \mathfrak{S}_4$	The normalizer of $\langle \beta_1 \rangle$ in $\mathbf{M}$.
β_2	An element of order 3 in A_4.
$S \cong \mathbb{Z}_3^2$	The group $\langle \beta_1, \beta_2 \rangle$.
ι	An involution in $S_4 \backslash A_4$ inverting β_2 and centralizing $\tau \in Z(\mathbf{R}_{1,2,1})$.
c	An involution in $\mathbf{M_c}.2 \backslash \mathbf{M_c}$ inverting $\mathbf{E}$ and centralizing $\mathbf{M}$.
$K \cong \mathbb{Z}_2^2$	The 2-core $O_2(A_4)$ of A_4, which coincides with $Z(\mathbf{R}_{1,2,1})$.
$K^{(1)} \cong \mathbb{Z}_2^2$	The four-group $\langle \iota, c \rangle$.

TABLE 4.5. Notation for the remainder of the chapter

We note the following for future use.

LEMMA 4.9.1. *The group* $S \rtimes K^{(1)}$ *is the direct product* $\langle \beta_1, c \rangle \times \langle \beta_2, \iota c \rangle$ *of two copies of* $\mathfrak{S}_3$. *Hence* $O_2(S \rtimes K^{(1)}) = \{1\}$.

PROOF. This is a consequence of the fact that c inverts β_1 and centralizes β_2, while ι inverts S. $\square$

LEMMA 4.9.2. $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1}) = \mathbf{R}_{1,2,1} \rtimes \big(\mathrm{S} \rtimes \mathrm{K}^{(1)}\big)$.

PROOF. The group $\mathrm{S} \rtimes \mathrm{K}^{(1)}$ normalizes $\mathrm{N}_{\mathbf{M_c}}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big)$, as it normalizes $\mathrm{K} = \mathrm{Z}(\mathbf{R}_{1,2,1})$ and $\mathbf{M_c}$. By Lemmas 4.3.14 and 4.4.2 it follows that it normalizes $\mathbf{R}_{1,2,1} = \mathrm{O}_2\big(\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})\big) = \mathrm{O}_2\big(\mathrm{N}_{\mathbf{M_c}}(\mathrm{Z}(\mathbf{R}_{1,2,1}))\big)$. Hence $(\mathrm{S} \rtimes \mathrm{K}^{(1)}) \cap \mathbf{R}_{1,2,1}$ is contained in $\mathrm{O}_2(\mathrm{S} \rtimes \mathrm{K}^{(1)})$, which is $\{1\}$ by Lemma 4.9.1.

The group $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})/\mathbf{R}_{1,2,1}$ has order $|3^2\colon 2|\cdot 2 = 36$, by Lemma 4.4.2. This is also the order of $\mathrm{S} \rtimes \mathrm{K}^{(1)}$. We conclude that $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1}) = \mathbf{R}_{1,2,1} \rtimes \big(\mathrm{S} \rtimes \mathrm{K}^{(1)}\big)$. $\square$

COROLLARY 4.9.3. $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})/\mathbf{R}_{1,2,1}$ *acts faithfully on* V.

PROOF. We have $\mathrm{V} = \mathbf{R}_{1,2,1}/\,\Phi(\mathbf{R}_{1,2,1})$, by Corollary 4.3.15. Then [**As86**, 24.1] implies that $\mathrm{C}_{\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})}(\mathrm{V})$ is a 2-group. So $\mathrm{C}_{\mathrm{S}\rtimes\mathrm{K}^{(1)}}(\mathrm{V})$ is a normal 2-subgroup of $\mathrm{S} \rtimes \mathrm{K}^{(1)}$. But $\mathrm{O}_2(\mathrm{S} \rtimes \mathrm{K}^{(1)}) = \{1\}$, by Lemma 4.9.1. The result follows. $\square$

COROLLARY 4.9.4. $\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) = \mathbf{R}_{1,2,1} \rtimes \big(\langle \beta_1, c\rangle \times \langle \iota c\rangle\big) \cong \mathbf{R}_{1,2,1} \rtimes \big(\mathfrak{S}_3 \times \mathbb{Z}_2\big)$ *and* $\mathrm{C}_{\mathbf{M_c}.2}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big) = \mathbf{R}_{1,2,1} \rtimes \langle \beta_1, c\rangle \cong \mathbf{R}_{1,2,1} \rtimes \mathfrak{S}_3$.

PROOF. By Lemma 4.3.14, we know that $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ acts as the full automorphism group on $\mathrm{Z}(\mathbf{R}_{1,2,1})$. In particular $\mathrm{S} \rtimes \mathrm{K}^{(1)}$ acts doubly transitively on $\mathrm{K}^{\#} = \mathrm{Z}(\mathbf{R}_{1,2,1})^{\#}$.

The group $\langle \beta_1, \mathrm{K}^{(1)}\rangle$ is a subgroup of index 3 in $\mathrm{S} \rtimes \mathrm{K}^{(1)}$ which centralizes $\tau \in \mathrm{K}$, since $\mathrm{K}^{(1)}$ centralizes τ and $\mathrm{K} \le \mathrm{A}_4 = \mathrm{C}_{\mathbf{M}}(\beta_1)$. Hence it must coincide with $\mathrm{C}_{\mathrm{S}\rtimes\mathrm{K}^{(1)}}(\tau)$. So $\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) = \mathrm{C}_{\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})}(\tau) = \mathbf{R}_{1,2,1} \rtimes \big(\langle \beta_1, c\rangle \times \langle \iota c\rangle\big)$.

The group $\langle \beta_1, c\rangle$ is a subgroup of index 2 in $\mathrm{C}_{\mathrm{S}\rtimes\mathrm{K}^{(1)}}(\tau)$ which centralizes K. Hence it must coincide with $\mathrm{C}_{\mathrm{S}\rtimes\mathrm{K}^{(1)}}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big)$. So $\mathrm{C}_{\mathbf{M_c}\cdot 2}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big) = \mathbf{R}_{1,2,1} \rtimes \langle \beta_1, c\rangle$. $\square$

Corollary 4.9.5. *If we identify* S *and* $\mathrm{K}^{(1)}$ *with their images in* $\mathrm{N}_{\mathbf{M_c}\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1})$*, then there exists a decomposition* $\mathrm{S} = \mathrm{Y}_1 \times \mathrm{Y}_2$ *of* S *into cyclic subgroups, such that* $\mathrm{N}_{\mathbf{M_c}\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1}) = \big(\mathrm{A_4}^{(1)} \times \mathrm{A_4}^{(2)}\big) \rtimes \mathrm{K}^{(1)}$*, where* $\mathrm{A_4}^{(i)} = \mathrm{V}_i \rtimes \mathrm{Y}_i \cong \mathfrak{A}_4$ *and* $\mathrm{A_4}^{(i)} \rtimes \langle \iota\rangle \cong \mathfrak{S}_4$*, for* $i = 1, 2$*. Moreover* c *transposes* $\mathrm{A_4}^{(1)}$ *and* $\mathrm{A_4}^{(2)}$*.*

Proof. We identify the group $\mathrm{S} \rtimes \langle \iota\rangle$ with its image in $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1})$. So this group acts faithfully on V. Both $\mathbf{R}_{3,1}$ and $\mathbf{R}_{1,3}$ are normal subgroups of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$, by Lemma 4.3.14. Hence V_1 and V_2 are subgroups of V stabilized by S. Since ι inverts all elements of S, the fact that S acts non-trivially on both V_1 and V_2 implies that ι acts faithfully on both V_1 and V_2.

Let Y_i be the kernel of the action of $\mathrm{S} \rtimes \langle \iota\rangle$ on V_{3-i}, for $i = 1, 2$. Then $\mathrm{Y}_1 \cap \mathrm{Y}_2 = \{1\}$. So each of these kernels has order 3 and $\mathrm{S} = \mathrm{Y}_1 \times \mathrm{Y}_2$.

Consider the involution c in $\mathrm{N}_{\mathbf{M_c}\cdot 2}(\mathbf{R}_{1,2,1})\backslash\mathbf{M_c}$. By Corollary 4.1.9 and Lemma 4.3.10, this element cannot normalize $\mathbf{R}_{3,1}$. So $\mathbf{R}_{3,1}^c$ is an elementary abelian subgroup of $\mathbf{R}_{1,2,1}$ of order 2^4 distinct from $\mathbf{R}_{3,1}$. It then follows from Corollary 4.3.12 that $\mathbf{R}_{3,1}^c = \mathbf{R}_{1,3}$. In the same way $\mathbf{R}_{1,3}^c = \mathbf{R}_{3,1}$. So c exchanges V_1 and V_2 in its conjugation action on V. Hence it also exchanges Y_1 and Y_2. Now S has four subgroups of order 3, namely $\langle\beta_1\rangle, \langle\beta_2\rangle, \langle\beta_1\beta_2\rangle$ and $\langle\beta_1\beta_2^{-1}\rangle$. Since c inverts β_1 and centralizes β_2, we see that it transposes $\langle\beta_1\beta_2\rangle$ and $\langle\beta_1\beta_2^{-1}\rangle$, while fixing $\langle\beta_1\rangle$ and $\langle\beta_2\rangle$. Hence Y_1 is one of the subgroups $\langle \beta_1\beta_2 \rangle$ and $\langle \beta_1\beta_2^{-1} \rangle$, while Y_2 is the other. We set ${\mathrm{A}_4}^{(1)} = \mathrm{V}_1 \rtimes \mathrm{Y}_1$ and ${\mathrm{A}_4}^{(2)} = \mathrm{V}_2 \rtimes \mathrm{Y}_2$.

Since Y_i acts faithfully on V_i, it follows that ${\mathrm{A}_4}^{(i)} \cong \mathfrak{A}_4$, for $i = 1, 2$. It is also clear that ${\mathrm{A}_4}^{(i)} \rtimes \langle \iota \rangle \cong \mathfrak{S}_4$, while c transposes ${\mathrm{A}_4}^{(1)}$ and ${\mathrm{A}_4}^{(2)}$. $\square$

We can now prove the following

PROPOSITION 4.9.6. $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})/\mathbf{T}\big) = \{4^4, 8^2\}$ *and* $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) \mid \mathbf{T}\big) = \{8^2, 16\}$. *Hence*

$$\mathrm{Inv}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})\big) = \{1^2, 2, 3^2, 4^2, 6, 8^2\}, \textit{ and}$$

$$\mathrm{InvDef}_2\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})\big) = \{7^4, 6^2, 5^2, 4^2\}.$$

Thus

$$\begin{aligned} &\mathrm{k}(\, C_{12}, \mathsf{B}_0, 7, \overline{\mathbf{M_c}.2}\,) = 4, \quad \mathrm{k}(\, C_{12}, \mathsf{B}_0, 6, \overline{\mathbf{M_c}.2}\,) = 2,\\ &\mathrm{k}(\, C_{12}, \mathsf{B}_0, 5, \overline{\mathbf{M_c}.2}\,) = 2, \quad \mathrm{k}(\, C_{12}, \mathsf{B}_0, 4, \overline{\mathbf{M_c}.2}\,) = 2,\\ &\mathrm{k}(\, C_{12}, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = 0, \quad \textit{for all other values of } d. \end{aligned} \tag{4.9.7}$$

PROOF. From the list of maximal subgroups of $\mathfrak{A}_8$ in the Atlas, $\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1})$ is isomorphic to $2.\big(\mathbb{Z}_2 \wr \mathfrak{S}_4\big)$. We compute the character degrees of $\mathbb{Z}_2 \wr \mathfrak{S}_4$ by using Clifford Theory on the linear characters of the base group $\mathbb{Z}_2^4$. Fixing some ordered basis of the base group $\mathbb{Z}_2^4$, we let (i_1, i_2, i_3, i_4) denote the character of $\mathbb{Z}_2^4$ sending the j^{th} basis element to $(-1)^{i_j}$, for each $j = 1, 2, 3, 4$. Table 4.6 on the following page summarizes our results. It follows from this table that $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1})/\mathbf{T}\big) = \{1^4, 2^2, 3^4, 4^4, 6^4, 8^2\}$. Comparing this with (4.7.7), we see that

$$\mathrm{Inv}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})/\mathbf{T}\big) = \{1^2, 2, 3^2, 4^2, 6, 8\}. \tag{4.9.8}$$

Base Character	Orbit Length	Stabilizer Structure	Stabilizer Character Degrees	Induced Character Degrees
$(1,1,1,1)$	1	$\mathfrak{S}_4$	$1^2, 2, 3^2$	$1^2, 2, 3^2$
$(1,1,1,0)$	4	$\mathfrak{S}_3 \times \mathfrak{S}_1$	$1^2, 2$	$4^2, 8$
$(1,1,0,0)$	6	$\mathfrak{S}_2 \times \mathfrak{S}_2$	1^4	6^4
$(1,0,0,0)$	4	$\mathfrak{S}_1 \times \mathfrak{S}_3$	$1^2, 2$	$4^2, 8$
$(0,0,0,0)$	1	$\mathfrak{S}_4$	$1^2, 2, 3^2$	$1^2, 2, 3^2$

TABLE 4.6. The Irreducible characters of $\mathbb{Z}_2 \wr \mathfrak{S}_4$

The group $\mathrm{Z}(\mathbf{R}_{1,2,1})/\mathbf{T}$ is the diagonal embedding of $\mathbb{Z}_2$ in the base subgroup $\mathbb{Z}_2^4$ of $\mathbb{Z}_2 \wr \mathfrak{S}_4$. The characters $(0,0,0,0)$, $(1,1,0,0)$, $(1,1,1,1)$ are all trivial on this subgroup, while $(1,0,0,0)$ and $(1,1,1,0)$ are not. Hence $\mathrm{Irr}\big(\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})/\mathbf{T}\big)$ consists of those irreducible characters of $\mathbb{Z}_2 \wr \mathfrak{S}_4$ lying over the latter two characters. So from the table above, $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})/\mathbf{T}\big) = \{4^4, 8^2\}$.

By Lemma 4.7.1 the group $\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1})$ acts transitively on the two elements of $\mathrm{Irr}(\mathbf{R}_{1,2,1} \mid \mathbf{T})$. Moreover the stabilizer of an element of $\mathrm{Irr}(\mathbf{R}_{1,2,1} \mid \mathbf{T})$ in $\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1})$ is $\mathrm{C}_{\mathbf{M}_c.2}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big)$. From Corollary 4.9.4 the group $\mathrm{C}_{\mathbf{M}_c.2}\big(\mathrm{Z}(\mathbf{R}_{1,2,1})\big)$ is of the form $\mathbf{R}_{1,2,1} \rtimes \mathfrak{S}_3$. Since all Sylow subgroups of $\mathfrak{S}_3$ are cyclic, we immediately obtain from Clifford Theory and Lemma 4.7.1 that $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) \mid \mathbf{T}\big) = \{8^2, 16\}$. Comparing this with (4.7.4) we deduce that

$$\mathrm{Inv}\big(\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) \mid \mathbf{T}\big) = \{8\}. \tag{4.9.9}$$

We obtain $\operatorname{Inv}\big(\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})\big)$ from (4.9.8) and (4.9.9). By Proposition 4.7.6, the group $\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1}) = \mathrm{N}_{\mathbf{M_c}}(C_{12})$ has a unique 2-block, which necessarily induces the principal 2-block, B_0, of $\mathbf{M_c}$. The rest of the proposition now follows. □

LEMMA 4.9.10. $\operatorname{Deg}\big(\,\mathrm{N}_{\mathbf{M_c}\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1})\,\big) = \{1^4, 2^4, 4, 6^2, 9^4, 12\}$. *Hence*

$$\begin{aligned}&\operatorname{Inv}\big(\,\mathrm{N}_{\mathbf{M_c}\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1})\,\big) = \{1^2, 2^2, 9^2\}, \text{ and}\\ &\operatorname{InvDef}_2\big(\,\mathrm{N}_{\mathbf{M_c}\cdot 2}(\mathbf{R}_{1,2,1}) \bmod \mathrm{Z}(\mathbf{R}_{1,2,1})\,\big) = \{7^4, 6^2\}.\end{aligned} \tag{4.9.11}$$

PROOF. We use the notation of Corollary 4.9.5. So $\mathrm{N}_{\mathbf{M_c}\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1}) = \big({\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}\big) \rtimes \mathrm{K}^{(1)}$, where ${\mathrm{A}_4}^{(i)} \cong \mathfrak{A}_4$ for $i = 1, 2$. Let $\lambda_1^i, \lambda_2^i, \lambda_3^i$ be the distinct linear characters of ${\mathrm{A}_4}^{(i)}$, with λ_1^i the trivial character. Let μ^i denote the single irreducible character of ${\mathrm{A}_4}^{(i)}$ of degree 3.

Then

$$\lambda_1^1 \times \mu^2,\ \ \mu^1 \times \lambda_1^2,\ \ \lambda_2^1 \times \mu^2,\ \ \mu^1 \times \lambda_2^2,\ \ \lambda_3^1 \times \mu^2,\ \ \mu^1 \times \lambda_3^2,\ \ \mu^1 \times \mu^2,$$

is a complete list of those characters of ${\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}$ which are non-trivial on V.

The determinantal character $\det(\mu^i)$ of μ^i, as given in Definition 1.2.12, satisfies $\overline{\det(\mu^i)} = \det(\overline{\mu^i}) = \det(\mu^i)$, for $i = 1, 2$. So $\det(\mu^i)$ is the trivial character of ${\mathrm{A}_4}^{(i)}$. Hence $\det(\mu^1 \times \mu^2) = \det(\mu^1) \times \det(\mu^2)$ is the trivial character of ${\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}$. It follows that the determinantal order $\mathrm{o}(\mu^1 \times \mu^2)$, as given in Definition 1.2.13, is 1.

The unique character $\mu^1 \times \mu^2$ of ${\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}$ of degree 9 is necessarily $\mathrm{K}^{(1)}$-invariant. Then, since

$$\Big(\ \big|({\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}) \rtimes \mathrm{K}^{(1)} \colon {\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}\big|\ ,\ \mathrm{o}(\mu^1 \times \mu^2)\big(\mu^1 \times \mu^2\big)(1)\ \Big) = (4, 9) = 1,$$

Theorem 1.2.14 guarantees that $\mu^1 \times \mu^2$ extends to $\big({\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}\big) \rtimes \mathrm{K}^{(1)}$. Hence

$$\mathrm{Deg}\big(\mathrm{N}_{\mathbf{M}_c\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1}) \mid \mu^1 \times \mu^2\big) = \{9^4\}. \tag{4.9.12}$$

Recall from Corollary 4.9.5 that ${\mathrm{A}_4}^{(i)} \rtimes \langle \iota \rangle \cong \mathfrak{S}_4$, for $i = 1, 2$, and that c interchanges ${\mathrm{A}_4}^{(1)}$ and ${\mathrm{A}_4}^{(2)}$. We now examine the action of $\mathrm{K}^{(1)} = \langle \iota, c \rangle$ on $\mathrm{Irr}({\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)})$.

The characters $\lambda_1^1 \times \mu^2$ and $\mu^1 \times \lambda_1^2$ are stabilized by ι and transposed by c. So they form a single $\mathrm{K}^{(1)}$-orbit. Thus

$$\mathrm{Deg}\big(\mathrm{N}_{\mathbf{M}_c\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1}) \mid \lambda_1^1 \times \mu^2\big) = \{6^2\}. \tag{4.9.13}$$

The stabilizer of $\lambda_2^1 \times \mu^2$ in $\mathrm{K}^{(1)}$ is trivial. Hence the remaining four characters $\lambda_2^1 \times \mu^2$, $\mu^1 \times \lambda_2^2$, $\lambda_3^1 \times \mu^2$, $\mu^1 \times \lambda_3^2$ of ${\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}$ form a single $\mathrm{K}^{(1)}$-orbit. So

$$\mathrm{Deg}\big(\mathrm{N}_{\mathbf{M}_c\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1}) \mid \lambda_2^1 \times \mu^2\big) = \{12\}. \tag{4.9.14}$$

The quotient $\Big(\big({\mathrm{A}_4}^{(1)} \times {\mathrm{A}_4}^{(2)}\big) \rtimes \mathrm{K}^{(1)}\Big)/\mathrm{V}$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_3$. Hence

$$\mathrm{Deg}\big(\mathrm{N}_{\mathbf{M}_c\cdot 2}(\mathbf{R}_{1,2,1})/\mathbf{R}_{1,2,1}\big) = \mathrm{Deg}(\mathfrak{S}_3 \times \mathfrak{S}_3) = \{1^4, 2^4, 4\}. \tag{4.9.15}$$

The multiset $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{M}_c\cdot 2}(\mathbf{R}_{1,2,1})/\,\mathrm{Z}(\mathbf{R}_{1,2,1})\big)$ can be obtained from (4.9.12), (4.9.13), (4.9.14) and (4.9.15). The rest of the lemma now follows from a comparison with Lemma 4.7.12. □

Lemma 4.9.16.

$$\mathrm{Inv}\big(\mathrm{N}_{\mathbf{M}_c}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big) = \{12^2, 24\},$$

$$\mathrm{InvDef}_2\big(\mathrm{N}_{\mathbf{M}_c}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big) = \{5^2, 4\}.$$

PROOF. From Lemma 4.9.2, we have $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})/\mathbf{R}_{1,2,1} \cong (\mathfrak{S}_3 \times \mathfrak{S}_3)$. By Lemma 4.7.1 there is a single $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})$-orbit of characters of $\mathbf{R}_{1,2,1}$ non-trivial on $\mathrm{Z}(\mathbf{R}_{1,2,1})$. Moreover, a character χ of this orbit vanishes outside $\mathrm{Z}(\mathbf{R}_{1,2,1})$. We may suppose, without loss of generality, that $\mathrm{Ker}(\chi) = \mathbf{T}$. Then χ is invariant in $\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1})$. But $\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1})$ is a sugroup of index three in $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})$. So it must coincide with $\mathrm{I}_{\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})}(\chi)$. Now $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{H}.2}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})/\mathbf{T}\big) = \{4^4, 8^2\}$, by Proposition 4.9.6. Hence $\mathrm{Deg}\big(\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1}) \mid \mathrm{Z}(\mathbf{R}_{1,2,1})\big) = \{12^4, 24^2\}$.

The lemma now follows from a comparison with the result of Lemma 4.7.10. □

From the previous two lemmas we obtain the following

PROPOSITION 4.9.17.

$$\mathrm{Inv}\big(\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})\big) = \{1^2, 2^2, 9^2, 12^2, 24\},$$

$$\mathrm{InvDef}_2\big(\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})\big) = \{7^4, 6^2, 5^2, 4\}.$$

Hence

$$\begin{aligned} &\mathrm{k}(\,C_6, \mathsf{B}_0, 7, \overline{\mathbf{M_c}.2}\,) = 4, \quad \mathrm{k}(\,C_6, \mathsf{B}_0, 6, \overline{\mathbf{M_c}.2}\,) = 2, \\ &\mathrm{k}(\,C_6, \mathsf{B}_0, 5, \overline{\mathbf{M_c}.2}\,) = 2, \quad \mathrm{k}(\,C_6, \mathsf{B}_0, 4, \overline{\mathbf{M_c}.2}\,) = 1, \\ &\mathrm{k}(\,C_6, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) = 0, \quad \textit{for all other values of } d. \end{aligned} \tag{4.9.18}$$

PROOF. This follows at once from Lemmas 4.9.10 and 4.9.16 and the fact that $\mathrm{N}_{\mathbf{M_c}.2}(C_6)$= $\mathrm{N}_{\mathbf{M_c}.2}(\mathbf{R}_{1,2,1})$. □

We now have enough information for the following theorem.

THEOREM 4.9.19. *The Invariant Conjecture holds for McLaughlin's simple group and the prime $p = 2$.*

PROOF. By [**Da96**] we need only consider the unique 2-block, B_0, of $\mathbf{M_c}$ having a non-cyclic defect group. In view of Table 4.3 on page 110, Conjecture 1.4.4 for the block B_0 is equivalent to the equation

$$\begin{aligned} \mathrm{k}(\,C_1, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) + \mathrm{k}(\,C_{12}, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) &= \\ &\mathrm{k}(\,C_6, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,) + \mathrm{k}(\,C_7, \mathsf{B}_0, d, \overline{\mathbf{M_c}.2}\,), \end{aligned} \tag{4.9.20}$$

for all values of $d \in \mathbb{Z}$.

From (4.8.2), (4.9.7), (4.9.18) and (4.5.2) we obtain the following sums for the equation above for various values of d:

2-Defect	C_1		C_{12}		C_6		C_7
7	4	+	4	=	4	+	4
6	2	+	2	=	2	+	2
5	2	+	2	=	2	+	2
4	1	+	2	=	1	+	2
3	1	+	0	=	0	+	1

TABLE 4.7. The Invariant Conjecture for $p = 2$

The summands in Equation (4.9.20) are zero for all other values of d. This completes the proof. $\square$

4.10. The Projective conjecture for the prime $p = 2$

By Theorem 3.12.8, the McLaughlin group has a cyclic Schur multiplier of order 3. As in Section 3.12, we let $\mathbf{A}$ denote the center of the universal perfect covering group $\widehat{\mathbf{M_c}}$ of $\mathbf{M_c}$. So $\mathbf{A}$ is cyclic of order 3. If X is a subgroup of $\mathbf{M_c}$, then we will denote its inverse image in $\widehat{\mathbf{M_c}}$ by $\mathbf{A}$.X or $\widehat{\mathrm{X}}$.

Let ρ be some fixed non-trivial character of $\mathbf{A}$.

LEMMA 4.10.1. *$\widehat{\mathbf{M_c}}$ has three 2-blocks, $\mathsf{B}_0^*, \mathsf{B}_1^*$ and B_2^*, lying over the 2-block of $\mathbf{A}$ containing ρ. The block B_0^* has defect 7 and contains 18 characters, while B_1^* has defect 1 and contains 2 characters, and B_2^* has defect 0.*

PROOF. We use the Atlas notation for the characters in $\mathrm{Irr}(\widehat{\mathbf{M_c}} \mid \rho)$. Two of these characters, χ_{36} and χ_{37}, have defect 1 and lie in a 2-block B_1^* of defect 1, while another, χ_{41}, lies in a 2-block B_2^* of defect 0. The remaining 18 characters lie in a single 2-block, B_0^*, of defect 7. □

We list the irreducible characters of B_0^* and their 2-defects:

Character	χ_{25}	χ_{26}	χ_{27}	χ_{28}	χ_{29}	χ_{30}	χ_{31}	χ_{32}	χ_{33}
Degree	126	126	792	1980	2376	2376	2520	2520	2772
Defect	6	6	4	5	4	4	4	4	5
Character	χ_{34}	χ_{35}	χ_{38}	χ_{39}	χ_{40}	χ_{42}	χ_{43}	χ_{44}	χ_{45}
Degree	4752	5103	7875	8019	8019	10395	10395	10395	12375
Defect	3	7	7	7	7	7	7	7	7

Thus

$$\text{(4.10.2)}\qquad \begin{aligned} &\mathrm{k}(C_1,\mathsf{B}_0^*,7\mid\rho)=8, && \mathrm{k}(C_1,\mathsf{B}_0^*,6\mid\rho)=2,\\ &\mathrm{k}(C_1,\mathsf{B}_0^*,5\mid\rho)=2, && \mathrm{k}(C_1,\mathsf{B}_0^*,4\mid\rho)=5,\\ &\mathrm{k}(C_1,\mathsf{B}_0^*,3\mid\rho)=1, && \mathrm{k}(C_1,\mathsf{B}_0^*,d\mid\rho)=0, \quad \text{for all other values of } d. \end{aligned}$$

The covering group $\widehat{\mathbf{H}}$ of $\mathbf{H}\cong 2.\mathfrak{A}_8$ splits over $\mathbf{A}$, because $\mathbf{H}$ is the universal covering group of the alternating group $\mathfrak{A}_8$. So its subgroups $\mathbf{A}.\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{3,1})$, $\mathbf{A}.\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,3})$ and $\mathbf{A}.\mathrm{N}_{\mathbf{H}}(\mathbf{R}_{1,2,1})$ also split over $\mathbf{A}$. Now $\mathbf{H}$ has a unique 2-block of defect greater than 1, and by Propositions 4.6.2 and 4.7.6, each of its subgroups $\mathbf{R}_{3,1}$, $\mathbf{R}_{1,3}$ and $\mathbf{R}_{1,2,1}$ has a unique 2-block. Then in view of Table 4.3 on page 110, it follows that

$$\text{(4.10.3)}\qquad \mathrm{k}(C_i,\mathsf{B}_0^*,d\mid\rho)=\mathrm{k}(C_i,\mathsf{B}_0,d),\ \text{ for } 7\le i\le 12, \text{ and all values of } d.$$

Next we consider the groups $\mathbf{A}.\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$, $\mathbf{A}.\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3})$ and $\mathbf{A}.\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$. Recall from Table 4.5 on page 126 and from Lemma 4.9.2 that the Sylow 3-subgroup S of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})$ is elementary abelian of order 9. Using Lemma 4.3.14 and Corollary 4.3.11, we see that S is also a Sylow 3-subgroup of both $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})$ and $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,3})$.

Lemma 4.10.4. *The covering group $\widehat{\mathrm{S}}$ of S in $\widehat{\mathbf{M}_\mathbf{c}}$ does not split over $\mathbf{A}$. Hence it is extra-special of type 3_+^{1+2}.*

Proof. By construction $\mathrm{S}=\langle\beta_1,\beta_2\rangle$, where $\beta_1\in\mathbf{E}$ is an element of the $(3B)$ class of $\mathbf{M_c}$, and β_2 is a 3-element of $\mathrm{A}_4\le\mathbf{M}$. In particular $\mathrm{S}\not\le\mathbf{E}$.

Let $\hat{\beta}_1$ be an element of $\widehat{\mathbf{M}}_\mathbf{c}$ having β_1 as its image in $\mathbf{M}_\mathbf{c}$. From the Atlas, $\hat{\beta}_1$ is a 3-element of the unique conjugacy class of $\widehat{\mathbf{M}}_\mathbf{c}$ lying over the conjugacy class $(3B)$ of $\mathbf{M}_\mathbf{c}$. In particular $|\mathrm{C}_{\widehat{\mathbf{M}}_\mathbf{c}}(\hat{\beta}_1)| = |\mathrm{C}_{\mathbf{M}_\mathbf{c}}(\beta_1)|$.

It follows from Proposition 3.1.12 that $\mathrm{C}_{\mathbf{M}_\mathbf{c}}(\beta_1) = \mathbf{E} \rtimes \mathrm{A}_4$. This group normalizes $\mathbf{E}$ and has a Sylow 3-subgroup of order 3^5. Hence the group $\mathrm{C}_{\widehat{\mathbf{M}}_\mathbf{c}}(\hat{\beta}_1) \leq \mathbf{A}.\,\mathrm{C}_{\mathbf{M}_\mathbf{c}}(\beta_1)$ normalizes $\widehat{\mathbf{E}}$, and has a Sylow 3-subgroup of order 3^5. But $\widehat{\mathbf{E}}$ is isomorphic to $\mathbb{Z}_3^5$, by Proposition 3.13.1. Hence it centralizes $\hat{\beta}_1$. So $\widehat{\mathbf{E}}$ is the unique Sylow 3-subgroup of $\mathrm{C}_{\widehat{\mathbf{M}}_\mathbf{c}}(\hat{\beta}_1)$. Since $\widehat{\mathrm{S}}$ is a 3-group normalizing $\widehat{\mathbf{E}}$, but not contained in $\widehat{\mathbf{E}}$, we conclude that $\widehat{\mathrm{S}} \not\leq \mathrm{C}_{\widehat{\mathbf{M}}_\mathbf{c}}(\hat{\beta}_1)$. In particular $\widehat{\mathrm{S}}$ is not abelian. So $\widehat{\mathrm{S}}$ does not split over $\mathbf{A}$.

From the Atlas, any element of $\widehat{\mathbf{M}}_\mathbf{c}$ whose image in $\mathbf{M}_\mathbf{c}$ has order 3 itself has order 3. Hence $\widehat{\mathrm{S}}$ has exponent 3.

The previous two paragraphs show that $\widehat{\mathrm{S}}$ is a non-abelian 3-group of exponent 3 and order 3^3. The lemma follows. ☐

We now study $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(C_3) = \mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1})$.

Proposition 4.10.5.

$$\mathrm{Deg}\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1}) \mid \rho\big) = \{3^6, 6^2, 9^2, 12^2, 24\},$$

$$\mathrm{Def}_2\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1}) \mid \rho\big) = \{7^8, 6^2, 5^2, 4\}.$$

Hence

$$\begin{aligned} &\mathrm{k}(C_3, \mathsf{B}_0^*, 7 \mid \rho) = 8, \qquad \mathrm{k}(C_3, \mathsf{B}_0^*, 6 \mid \rho) = 2,\\ &\mathrm{k}(C_3, \mathsf{B}_0^*, 5 \mid \rho) = 2, \qquad \mathrm{k}(C_3, \mathsf{B}_0^*, 4 \mid \rho) = 1, \\ &\mathrm{k}(C_3, \mathsf{B}_0^*, 3 \mid \rho) = 0, \qquad \mathrm{k}(C_3, \mathsf{B}_0^*, d \mid \rho) = 0, \quad \textit{for all other values of } d. \end{aligned} \tag{4.10.6}$$

Also

(4.10.7) $\qquad \mathrm{k}(C_3, \mathsf{B}_0^*, d \mid \rho) = \mathrm{k}(C_5, \mathsf{B}_0^*, d \mid \rho) = \mathrm{k}(C_6, \mathsf{B}_0^*, d \mid \rho)$, *for all values of* d.

PROOF. From Lemma 4.4.2 the normalizer of $\mathbf{R}_{1,2,1}$ in $\mathbf{M_c}$ is of the form $\mathbf{R}_{1,2,1} \rtimes (3^2 : 2)$. Then from Lemma 4.10.4, its covering group $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1})$ in $\widehat{\mathbf{M}_\mathbf{c}}$ is of the form $\mathbf{R}_{1,2,1}^{(1)} \rtimes (3_+^{1+2} : 2)$, where $\mathbf{R}_{1,2,1}^{(1)}$ is the unique Sylow 2-subgroup of the inverse image $\widehat{\mathbf{R}}_{1,2,1}$ of $\mathbf{R}_{1,2,1}$ in $\widehat{\mathbf{M}_\mathbf{c}}$.

The Sylow 3-subgroup 3_+^{1+2}, of a complement $3_+^{1+2} : 2$ to $\mathbf{R}_{1,2,1}^{(1)}$ in $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1})$, has exactly one irreducible character μ lying over ρ. Now μ has degree 3 and is invariant in $3_+^{1+2} : 2$. So there are exactly two characters of $3_+^{1+2} : 2$ lying over μ, and each has degree 3. We inflate these to characters of $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1})$. In this way we obtain all irreducible characters of $\mathrm{Irr}\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1}) \mid \rho\big)$ whose kernels contain $\mathbf{R}_{1,2,1}^{(1)}$.

Let χ be a non-trivial irreducible character of $\mathbf{R}_{1,2,1}^{(1)}$. Then there is one extension $\tilde{\chi}$ of χ to $\widehat{\mathbf{R}}_{1,2,1} = \mathbf{A} \times \mathbf{R}_{1,2,1}^{(1)}$ which lies over ρ. The stabilizer $\mathrm{I}_{\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1})}(\tilde{\chi})$ of this extension is then just $\mathbf{A}.\,\mathrm{I}_{\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})}(\chi)$. By Corollary 4.7.13, we know that $\mathbf{A}.\,\mathrm{I}_{\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})}(\chi)/\big(\mathbf{A} \times \mathbf{R}_{1,2,1}^{(1)}\big)$ has a Sylow 3-subgroup of order 1 or 3. Also, a Sylow 2-subgroup of $\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1})/\big(\mathbf{A} \times \mathbf{R}_{1,2,1}^{(1)}\big)$ is cyclic of order 2. Hence $\tilde{\chi}$ extends to its entire stabilizer, by Theorem 1.2.11. We conclude that there is a degree preserving bijection between $\mathrm{Irr}\big(\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{1,2,1}) \mid \mathbf{R}_{1,2,1}\big)$ and the set of those characters of $\mathrm{Irr}\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1}) \mid \rho\big)$ whose restriction to $\mathbf{R}_{1,2,1}$ is non-trivial.

All elements of $\mathrm{Irr}\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1}) \mid \rho\big)$ belong to a single 2-block b_0^* of $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,2,1})$ lying over the block of $\mathbf{A}$ containing ρ. This block b_0^* has a defect group containing

$\mathbf{R}_{1,2,1}^{(1)}$. In particular, the defect of $(\mathsf{b}_0^*)^{\widehat{\mathbf{M}}_\mathbf{c}}$ is at least $\log_2(|\mathbf{R}_{1,2,1}|) = 6$. It then follows from Lemma 4.10.1 that $(\mathsf{b}_0^*)^{\widehat{\mathbf{M}}_\mathbf{c}} = \mathsf{B}_0^*$.

The character degrees and defects given in the statement of the proposition now follow from the first paragraph of the proof, Lemmas 4.7.10 and 4.7.12, and the fact that $\mathrm{Deg}(\mathrm{N}_{\mathbf{M}_\mathbf{c}}(\mathbf{R}_{1,2,1})/\mathbf{R}_{1,2,1}) = \mathrm{Deg}(3^2\colon 2) = \{1^2, 2^4\}$. □

Next we deal with $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(C_2) = \mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1})$.

PROPOSITION 4.10.8.

$$\mathrm{Deg}\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1}) \mid \rho\big) = \{6, 15^3, 21^2, 24^2, 45^2, 90, 105, 120\},$$

$$\mathrm{Def}_2\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1}) \mid \rho\big) = \{7^8, 6^2, 4^3\}.$$

Hence

$$\begin{array}{lll} & \mathrm{k}(C_2, \mathsf{B}_0^*, 7 \mid \rho) = 8, & \mathrm{k}(C_2, \mathsf{B}_0^*, 6 \mid \rho) = 2, \\ (4.10.9) & \mathrm{k}(C_2, \mathsf{B}_0^*, 5 \mid \rho) = 0, & \mathrm{k}(C_2, \mathsf{B}_0^*, 4 \mid \rho) = 3, \\ & \mathrm{k}(C_2, \mathsf{B}_0^*, 3 \mid \rho) = 0, & \mathrm{k}(C_2, \mathsf{B}_0^*, d \mid \rho) = 0, \quad \textit{for all other values of } d. \end{array}$$

Also

$$(4.10.10) \qquad \mathrm{k}(C_4, \mathsf{B}_0^*, d \mid \rho) = \mathrm{k}(C_2, \mathsf{B}_0^*, d \mid \rho), \quad \textit{for all values of } d.$$

PROOF. From Corollary 4.3.11 and Lemma 4.10.4, the group $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1})$ is of the form $\mathbf{R}_{3,1}^{(1)} \rtimes \widehat{\Lambda}_{3,1}$, where $\mathbf{R}_{3,1}^{(1)}$ is the unique Sylow 2-subgroup of the cover of $\mathbf{R}_{3,1}$ in $\widehat{\mathbf{M}}_\mathbf{c}$, and the complement $\widehat{\Lambda}_{3,1}$ is a non-split central extension of $\Lambda_{3,1} \cong \mathfrak{A}_7$ by the cyclic group $\mathbf{A}$ of order 3.

From the character table of $\mathfrak{A}_7$ in the Atlas, we obtain

$$\begin{aligned} \operatorname{Deg}(\widehat{\Lambda}_{3,1} \mid \rho) &= \{6, 15^2, 21^2, 24^2\}, \\ \operatorname{Def}_2(\widehat{\Lambda}_{3,1} \mid \rho) &= \{7^4, 6, 4^2\} \end{aligned} \tag{4.10.11}$$

Let χ be a non-trivial linear character of $\mathbf{R}_{3,1}$. By Lemma 4.6.8, the $\Lambda_{3,1}$-orbit of χ is $\operatorname{Irr}(\mathbf{R}_{3,1})^{\#}$, and $\mathrm{I}_{\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}3,1)}(\chi)$ is of the form $\mathbf{R}_{3,1} \rtimes \mathrm{GL}(3,2)$. We let $\hat{\chi}$ be a linear character of $\widehat{\mathbf{R}}_{3,1} = \mathbf{A} \times \mathbf{R}_{3,1}^{(1)}$ lying over ρ and non-trivial on $\mathbf{R}_{3,1}^{(1)}$. Then the stabilizer of $\hat{\chi}$ in $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1})$ is of the form $\mathbf{A}.(\mathbf{R}_{3,1} \rtimes \mathrm{GL}(3,2))$. But $\mathbf{A}.(\mathbf{R}_{3,1} \rtimes \mathrm{GL}(3,2))$ splits over $\mathbf{A}$, since a Sylow 3-subgroup of $\mathbf{R}_{3,1} \rtimes \mathrm{GL}(3,2)$ is cyclic of order 3, and is not central in its normalizer. Hence there is a degree preserving bijection between $\operatorname{Irr}\big(\mathrm{I}_{\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1})}(\chi) \mid \chi\big)$ and $\operatorname{Irr}\big(\mathrm{I}_{\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1})}(\hat{\chi}) \mid \hat{\chi}\big)$. We conclude that there is also a degree preserving bijection between $\operatorname{Irr}(\mathrm{N}_{\mathbf{M_c}}(\mathbf{R}_{3,1}) \mid \mathbf{R}_{3,1})$ and the set of those characters of $\operatorname{Irr}\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1}) \mid \rho\big)$ non-trivial on $\mathbf{R}_{3,1}$.

We now obtain $\operatorname{Deg}\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1}) \mid \rho\big)$ and $\operatorname{Def}_2\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1}) \mid \rho\big)$ using (4.6.9) and (4.10.11).

All elements of $\operatorname{Irr}\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1}) \mid \rho\big)$ belong to a single 2-block b_0^* of $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1})$ lying over the block of $\mathbf{A}$ containing ρ. This block b_0^* has a defect group containing $\mathbf{R}_{3,1}^{(1)}$. In particular, the defect of $(\mathsf{b}_0^*)^{\widehat{\mathbf{M}}_\mathbf{c}}$ is at least $\log_2(|\mathbf{R}_{3,1}|) = 4$. It then follows from Lemma 4.10.1 that $(\mathsf{b}_0^*)^{\widehat{\mathbf{M}}_\mathbf{c}} = \mathsf{B}_0^*$.

We can now compute (4.10.9) using the multiset $\operatorname{Def}_2\big(\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1}) \mid \rho\big)$ and the fact that $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(C_2) = \mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1})$. Equation (4.10.10) holds because $\mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{3,1}) \cong \mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(\mathbf{R}_{1,3}) = \mathrm{N}_{\widehat{\mathbf{M}}_\mathbf{c}}(C_4)$. $\square$

We can now prove the following

THEOREM 4.10.12. *The Projective Conjecture holds for McLaughlin's Simple Group* $\mathbf{M_c}$ *and the prime* $p = 2$.

PROOF. We need only consider the 2-blocks B_0^* and B_1^* of positive defect in Lemma 4.10.1. Since B_1^* has defect 1, it has a cyclic defect group. So Conjecture 1.4.6 holds for it by [**Da96**]. In view of Table 4.3 on page 110, Conjecture 1.4.6 for the block B_0^* is equivalent to the equation

$$\begin{aligned}&\mathrm{k}(C_1, \mathsf{B}_0^*, d \mid \rho) + \mathrm{k}(C_3, \mathsf{B}_0^*, d \mid \rho) + \mathrm{k}(C_8, \mathsf{B}_0^*, d \mid \rho) + \mathrm{k}(C_{10}, \mathsf{B}_0^*, d \mid \rho) = \\ &\mathrm{k}(C_2, \mathsf{B}_0^*, d \mid \rho) + \mathrm{k}(C_4, \mathsf{B}_0^*, d \mid \rho) + \mathrm{k}(C_7, \mathsf{B}_0^*, d \mid \rho) + \mathrm{k}(C_9, \mathsf{B}_0^*, d \mid \rho),\end{aligned} \tag{4.10.13}$$

for all values of $d \in \mathbb{Z}$.

We obtain Table 4.8 using Equations (4.10.2), (4.10.6), (4.10.9) and (4.10.10), as well as (4.5.1), (4.6.3), (4.7.8), (4.6.4) and (4.10.3).

2-Defect	C_1		C_3		C_8		C_{10}		C_2		C_4		C_7		C_9
7	8	+	8	+	8	+	8	=	8	+	8	+	8	+	8
6	2	+	2	+	2	+	2	=	2	+	2	+	2	+	2
5	2	+	2	+	0	+	0	=	0	+	0	+	2	+	2
4	5	+	1	+	6	+	6	=	3	+	3	+	8	+	4
3	1	+	0	+	0	+	0	=	0	+	0	+	1	+	0

TABLE 4.8. The Projective Conjecture for $p = 2$

The summands in the above equation are zero for all other values of d. This completes the proof. □

Bibliography

[AM95] A. Adem, R. James Milgram, *The Cohomology of the McLaughlin Group and some associated groups*, preprint, 1995.

[AL93] J. L. Alperin, *Local Representation Theory*, Cambridge Studies in Advanced Mathematics 11, Camb. Univ. Press, 1993.

[Al87] J. Alperin, *Weights for finite groups*, Proc. Symp. Pure Math. 47, 369–379, 1987.

[AF90] J. L. Alperin, P. Fong, *Weights for Symmetric and General Linear Groups*, J. Algebra 131, 2–22, 1990.

[As86] M. Aschbacher, *Finite group theory*, Cambridge Studies in Advanced Mathematics 10, Camb. Univ. Press, 1986.

[As94] M. Aschbacher, *Sporadic Groups*, Cambridge Tracts in Mathematics 104, Camb. Univ. Press, 1994.

[BM90] H. Blau, G. Michler, *Modular Representation Theory of Finite Groups with T.I. Sylow p-subgroups*, Trans. AMS 319 (2), June 1990.

[Con71] J. Conway, *Three Lectures on exceptional groups*, in M. B. Powell, G. Higman, Finite Simple groups, Academic Press, London, 215–47, 1971.

[Con85] J. Conway, R. Curtis, S. Norton, R. Parker, R. Wilson, *Atlas of Finite Groups*, Oxford University Press, New York, London, Oxford, 1985.

[CR81] C. Curtis, I. Reiner, *Methods of Representation Theory*, Volumes I, II, Wiley, New York, 1981.

[Da92] E. C. Dade, *Counting Characters in blocks, I*, Invent. Math. 109, 187–210, 1992.

[Da94] E. C. Dade, *Counting Characters in blocks, II*, J. Reine Angew. Math. 448, 97–190, 1994.

[DaPr] E. C. Dade, *Counting Characters in blocks, 2.9*, preprint.

[Da96] E. C. Dade, *Counting Characters in Blocks with Cyclic Defect Groups I*, J. Algebra 186, 934–969, 1996.

[Ft82] W. Feit, *The Representation Theory of Finite Groups*, North-Holland Math. Library, 1982.

[Fk73] L. Finkelstein, *The Maximal Subgroups of Conway's Group C_3 and McLaughlin's Group*, J. Algebra 25, 58–89, 1973.

[Go80] D. Gorenstein, *Finite Groups*, second edition, Chelsea Publishing Co., 1980.

[GL83] D. Gorenstein, R. Lyons, *Memoirs of AMS*, No. 276 ,Vol. 42, p55, 1983.

[Gr87] R. L. Greiss Jr., *The Schur multiplier of McLaughlin's simple group*, Arch. Math. Vol. 48, 31, 1987.

[Hg97] J. Huang, *Counting Characters in Blocks of M_{22}*, J. Algebra 191, 1–75, 1997.

[Hu67] B. Huppert, *Endliche Gruppen I*, Springer Verlag, Berlin Heidelberg New-York, 1967.

[Is76] I. Isaacs, *Character Theory of Finite Groups*, Academic Press, New York, 1976.

[Ko97] S. Kotlica, *Verification of Dade's Conjecture for Janko Group J_3*, J. Algebra 187, 579–619, 1997.

[KR89] Reinhard Knörr, Geoffrey R. Robinson, *Some remarks on a conjecture of Alperin*, J. London Math. Soc. (2) 39, 48–60, 1989.

[JW72] Z. Janko, S. K. Wong, *A characterization of the McLaughlin's Simple Group*, J. Algebra 20, 203–225, 1972.

[NT89] H. Nagao, Y. Tsushima, *Representations of Finite Groups*, Academic Press, New York, 1989.

[R93] D. J. S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Math. 80, second edition, Springer-Verlag, New York, 1993.

[Ro86] G. Robinson, *A remark on a conjecture of Alperin*, preprint, 1986.

[Rt95] J. Rotman, *Group Theory*, Graduate Texts in Math. 148, fourth edition, Springer-Verlag, New York, 1995.

[Sc10] J. Schur, *Über die Darstellung der Symmetrischen und der alternierenden Gruppe durch gebrochene lineare substitutionen*, J. Für die Reine und Angewandte Mathematik, 139, 155–250, 1910–11.

Curriculum Vitae

John Cyril Murray was born in Clonmel, Ireland on February 9, 1968. He received a Certificate in Visual Education from Letterkenny Regional Technical College in 1986, a B. Electronic Engineering from University College Dublin in 1990, a Diploma in Mathematical Science from University College Dublin in 1992 and an M.Sc. in Mathematics from University College Dublin in 1993. He will be starting a Forbairt Postdoctoral Fellowship in September 1997 and is a member of the IMS and AMS.

John and his wife Deirdre McGowan have two children, Clara and Seóirse.